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SYNTHESIS OF TIME-OPTIMAL CONTROL  
FOR SECOND-ORDER NONLINEAR SYSTEMS

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ABSTRACT

This report presents results on the synthesis of time-optimal control for a second-order nonlinear system. The problem is to determine the feedback control as a function of the state variables. This is equivalent to finding the switching locus, since the time-optimal control is a relay control. The nonlinear system is represented by a soft spring characterized by the Duffing equation with negative nonlinear term.

The domain of controllability is described. For the case that the coefficient of the nonlinear term is sufficiently small a procedure is developed which determines the maximum number of switchings. Also the influence of damping is considered.

An analytic expression of the feedback control is derived for the conservative case. This expression contains a hyper-elliptic integral of the first kind. Since this integral cannot be solved in closed form estimates are derived to show the existence or non-existence of a switching point in certain regions of the domain of controllability.

## CHAPTER I

### INTRODUCTION

This thesis makes extensive reference to work done by Markus and Lee [1]. The principles and concepts which they developed needed in the following chapters will be presented in this chapter.

The system under consideration is

$$\ddot{x} + F(x, \dot{x}) = u$$

or

$$\dot{x} = y$$

$$\dot{y} = -F(x, y) + u \quad (1)$$

with  $F(0,0) = 0$ ,  $F(x,y)$  in the class  $C^1$  in the real number plane  $R^2$ ,  $u$  is in the compact interval  $\Omega$ :  $-1 \leq u \leq 1$ .

Most of the investigation in Chapters II through IV considers a conservative system for which  $\frac{\partial F(x,y)}{\partial y} = F_y(x,y) \geq 0$ . Then in equation (1)  $F(x,y)$  is replaced by  $f(x)$ . But any physical system has damping,  $F_y(x,y) \not\geq 0$ , and in section 3.2 a certain influence of the damping is discussed. Therefore in this chapter the theorems and definitions are stated for systems of the form (1) with nonzero  $F_y(x,y)$ .

#### 1.1 Controllability

An important concept is controllability which is defined for nonlinear systems in [1, Chapter 6.1]. This definition is stated here for second-order systems.

Definition: The domain  $G$  of null controllability for the control process (1) in  $\mathbb{R}^2$  and  $u$  in  $\Omega$  is defined as the set of all initial points  $(x_0, y_0) \in \mathbb{R}^2$ , each of which can be steered to the origin by some bounded measurable controller  $u(t) \in \Omega$  in some finite time duration. If  $G$  contains an open neighborhood of the origin, then the system is said to be locally controllable near the origin.

$G$  is connected because each point in  $G$  is joined to the origin by a continuous solution curve lying entirely in  $G$ .  $G$  is open in  $\mathbb{R}^2$  if and only if it contains a neighborhood of the origin, because the solutions of this differential equation depend continuously on the initial conditions.

In [1, Chapter 6.1] a theorem is proved which states a condition under which a system is locally controllable. This condition for the second-order system under consideration is that a certain matrix has rank 2.

For system (1) this matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & F_y(0,0) \end{bmatrix}.$$

Thus for this system independent of whether damping is present or not the domain  $G$  of controllability is an open connected subset of  $\mathbb{R}^3$  containing the origin.

### 1.2 Existence of Optimal Control

Another theorem needed for further discussion is the theorem on the existence of an optimal control [1, Chapter 4.2]. Specified for second-order autonomous systems it reads:

Theorem 1.2 Consider the nonlinear process in  $\mathbb{R}$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -F(x, y) + u \end{aligned} \tag{1}$$

where  $F(x,y)$  is  $C^1$  in  $R^2$ ,  $u \in \Omega$ , with the following conditions:

- (1) The sets of all initial points  $(x_0, y_0)$  and all target points  $(x_1, y_1)$  are nonempty and compact.
- (2) The control restraint interval  $\Omega$  is nonempty and compact.
- (3) The state constraints  $h^i(x,y) \geq 0$ ,  $i=1, \dots, r$ , if they exist, belong to the class  $C$  in  $R^2$ .
- (4) The family  $J$  of admissible controllers consists of all measurable functions  $u(t)$  on various time intervals  $[t_0, t_1] \subset [\tau_0, \tau_1]$  such that each  $u(t)$  has a response  $(x(t), y(t))$  on  $[t_0, t_1]$  steering  $(x(t_0), y(t_0)) = (x_0, y_0)$  to  $(x(t_1), y(t_1)) = (x_1, y_1)$  and  $u(t) \in \Omega$ ,  $h^i(x,y) \geq 0$ ,  $i=1, \dots, r$ .
- (5) The cost for each  $u \in J$  is

$$C(u) = g(x_1, y_1) + \int_{t_0}^{t_1} r^0(x(t), y(t), u(t)) dt + \max_{t_0 \leq t \leq t_1} \gamma(x(t), y(t))$$

where  $r^0$  is  $C^1$  in  $R^3$ ,  $g(x,y)$  and  $\gamma(x,y)$  are continuous in  $R^2$ .

If

- (a)  $J$  is nonempty.
- (b) There exists a uniform bound  $x^2(t) + y^2(t) \leq M$  on  $[t_0, t_1]$  for all responses  $(x(t), y(t))$  to  $u \in J$ .
- (c) The set  $\{r^0(x,y,u), \dot{x}, \dot{y} \mid u \in \Omega\}$  is convex in  $R^3$  for each  $(x,y)$ , then there exists an optimal controller  $u(t)$  on  $[t_0, t_1]$  in  $J$  minimizing  $C(u)$ .

For the time-optimal problem another class of controllers is defined.

Definition:  $\Delta$  is the class of all measurable controllers  $u(t) \in \Omega$  on finite time intervals  $[0, t_1]$  such that the response  $(x(t), y(t))$  of

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(1) defined on  $[0, t_1]$  with the given initial point  $(x(0), y(0))$  first reaches the origin at  $t = t_1$ .

## 1.3 The Optimal Control

The goal of Markus - Lee's research was to synthesize the time-optimal feedback controller for system (1) with certain conditions on  $F(x, y)$ .

A controller  $u(t)$  on a finite time interval  $[0, t_1]$  in  $\mathbb{R}$  is called time-optimal if for each  $u'(t)$  in  $\mathbb{R}$  on  $[0, t_1]$  one finds that  $t_1 \leq t_1'$ . By the maximum principle [1, Chapter 5] this optimal controller is a maximal controller. That means the following conditions must hold:

- (a) There exists an absolutely continuous, nowhere-vanishing adjoint vector  $(\eta_1(t), \eta_2(t))$  on  $[0, t_1]$  such that  $(x(t), y(t))$ ,  $(\eta_1(t), \eta_2(t))$  and  $u(t)$  satisfy the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial \eta_1} \quad \dot{\eta}_1 = -\frac{\partial H}{\partial y}$$

$$\dot{y} = \frac{\partial H}{\partial \eta_2} \quad \dot{\eta}_2 = -\frac{\partial H}{\partial x}$$

- (b) The Hamiltonian

$$H(\eta_1, \eta_2, x, y, u) = \eta_1 y - \eta_2 F(x, y) + \eta_2 u$$

is maximized with respect to  $u$  for almost all  $t$  on  $[0, t_1]$ .

- (c) The Hamiltonian is greater than or equal to zero and constant for all  $t$  on  $[0, t_1]$  along an optimal trajectory.

$$u(t) = \operatorname{sgn} \eta_2(t) = \begin{cases} -1 & \text{if } \eta_2(t) < 0 \\ 0 & \text{if } \eta_2(t) = 0 \\ 1 & \text{if } \eta_2(t) > 0. \end{cases}$$

That is, the time-optimal controller is a relay controller.

Definition: A controller  $u(t)$  on  $[0, t_1]$  in  $\Delta$  is called a relay controller if there exists a finite number of switching times

$$0 = \tau_0 < \tau_1 < \dots < \tau_k = t_1$$

such that on each open interval  $(\tau_{i-1}, \tau_i)$ ,  $i=1, \dots, k$ ,  $u(t)$  is constant and equals +1 or -1, and  $u(t)$  switches values on successive intervals.

Writing the Hamiltonian system for system (1) one obtains with the maximal controller

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -F(x, y) + 1 \quad \text{for } u = +1 \end{aligned} \tag{1_+}$$

or

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -F(x, y) - 1 \quad \text{for } u = -1 \end{aligned} \tag{1_-}$$

and

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 F_x(x, y) \\ \dot{\eta}_2 &= -\eta_1 + \eta_2 F_y(x, y) \end{aligned} \tag{2}$$

which is the adjoint system. Note that the Hamiltonian system, (1<sub>+</sub>) and (2) or (1<sub>-</sub>) and (2), is homogeneous. Since  $(\eta_1(t), \eta_2(t))$  is a nonvanishing vector solution in  $C^1$  on  $[0, t_1]$  of (2)  $\eta_2(t)$  has only a finite number of zeros which are simple.

By this maximum principle the control  $u(t)$  is only determined as a function of time. The feedback controller is a function of  $x$  and  $y$  which solves the synthesis problem.

#### 1.4 The Switching Locus

The following theorem referred to as the "Theorem of Interlacing Zeros" is very basic in establishing the properties of the switching

locus for the construction of the feedback controller [1, Chapter 7.1].

Theorem 1.4: Let  $u(t)$  for  $t \in [0, t_1]$  in  $\Delta$  be a maximal controller for system (1), and let  $(x(t), y(t))$  and  $(\eta_1(t), \eta_2(t))$  be the corresponding response of the Hamiltonian system, (1) and (2), on  $[0, t_1]$ . Let  $\tau_1, \tau_2$  be time on  $0 \leq \tau_1 < \tau_2 < t_1$ , then four assertions hold:

- (1) If  $\eta_2(\tau_1) = \eta_2(\tau_2) = 0$  and if  $y(\tau_1) = 0$ , then  $y(\tau_2) = 0$ .
- (2) If  $\eta_2(\tau_1) = \eta_2(\tau_2) = 0$  and if  $y(\tau_1) \neq 0$  then  $y(\tau_2) \neq 0$ , but there is a zero of  $y(t)$  on  $(\tau_1, \tau_2)$ .
- (3) If  $y(\tau_1) = y(\tau_2) = 0$ ,  $y(t) \neq 0$  on  $(\tau_1, \tau_2)$ , and if  $\eta_2(\tau_1) = 0$  then  $\eta_2(\tau_2) = 0$ .
- (4) If  $y(\tau_1) = y(\tau_2) = 0$ ,  $y(t) \neq 0$  on  $(\tau_1, \tau_2)$ , and if  $\eta_2(\tau_1) \neq 0$  then  $\eta_2(\tau_2) \neq 0$ , but there is a zero of  $\eta_2(t)$  on  $(\tau_1, \tau_2)$ .

In other words: Provided the zeros of  $y(t)$  are isolated, they either coincide with the zeros of  $\eta_2(t)$ , or no zero of  $y(t)$  is a zero of  $\eta_2(t)$ , but these two sets of zeros are interlaced.

Proof: Assume  $\eta_2(\tau_1) = \eta_2(\tau_2) = 0$

The Hamiltonian is  $\eta_1 y - \eta_2 F(x, y) + |\eta_2| = \text{const.} \geq 0$  on  $[0, t_1]$

Therefore  $\eta_1(\tau_1) y(\tau_1) = \eta_1(\tau_2) y(\tau_2) \geq 0$ . Since the zeros of  $\eta_2(t)$  are simple and by equation (2)  $\eta_1(\tau_1) \eta_1(\tau_2) < 0$ . Thus either  $y(\tau_1) = 0$  if and only if  $y(\tau_2) = 0$ , or if  $y(\tau_1) \neq 0$ , then  $y(\tau_1) y(\tau_2) < 0$ . So there is a zero of  $y(t)$  on  $(\tau_1, \tau_2)$ . This proves assertions (1) and (2).

Now assume  $y(\tau_1) = y(\tau_2) = 0$  and  $y(t) \neq 0$  on  $(\tau_1, \tau_2)$ . The condition that the Hamiltonian is constant on  $[0, t_1]$  yields

$$\dot{y}(\tau_1) \eta_2(\tau_1) = \dot{y}(\tau_2) \eta_2(\tau_2)$$

With  $y(t) \neq 0$  on  $(\tau_1, \tau_2)$  the uniqueness property of differential equations requires  $y^2(t) + \dot{y}^2(t) \neq 0$  on  $[\tau_1, \tau_2]$ . Therefore if  $\eta_2(\tau_1) = 0$  then

$\eta_2(\tau_2) = 0$ . If  $y(\tau_1) = y(\tau_2) = 0$ ,  $y(t) \neq 0$  on  $(\tau_1, \tau_2)$  and  $\eta_2(\tau_1) \neq 0$  then  $\eta_2(\tau_2) \neq 0$  by the above equation. If  $\eta_2(t)$  vanishes nowhere on  $(\tau_1, \tau_2)$  then  $\dot{y}(\tau_1) \dot{y}(\tau_2) > 0$  which is impossible since  $\tau_1$  and  $\tau_2$  are consecutive zeros of  $y(t)$ . This completes the proof.

Corollary 1.4: It is easy to show that at a switching point:  $\eta_2(\tau) = 0$

$$\text{if } y(\tau) > 0 \quad \text{then } \dot{\eta}_2(\tau) < 0$$

$$\text{and if } y(\tau) < 0 \quad \text{then } \dot{\eta}_2(\tau) > 0$$

At  $\tau$  the second equation of the adjoint system (2) is

$$\dot{\eta}_2(\tau) = -\eta_1(\tau)$$

and the Hamiltonian is

$$\eta_1(\tau) y(\tau) \geq 0$$

$$\text{or } \eta_2(\tau) y(\tau) \leq 0$$

Therefore  $u(t)$  switches from +1 to -1 in  $y \geq 0$  and from -1 to +1 in  $y \leq 0$ , going forward in time along an optimal trajectory.

Markus and Less describe the domain of controllability and prove the existence of an optimal control for the cases

(a) attractive force,  $x F(x,0) > 0$  for all  $x \neq 0$

(b) repulsive force  $F_x(x,0) < 0$  for all  $x \neq 0$

with nonnegative damping  $F_y(x,y) \geq 0$  in  $R^2$ . A similar theorem with proof for a different condition on  $F(x,y)$  will be presented in Chapter II.

The synthesis problem is solved by determining the switching locus. For a second order system it may be described as a function of  $x$ :  $W(x)$ .

Definition: The switching locus  $W$  consists of the points  $(x,y)$  at which the maximal responses switch from the solution family of  $(l_+)$  to  $(l_-)$  or vice versa. The origin is included in  $W$ .

Therefore the trajectory through the origin (or a segment of it) must belong to  $W$ . This indicates a way for the construction of  $W$ . Due to the theorem of interlacing zeros and its corollary a part of the solution of  $(l_+)$  through the origin in the fourth quadrant is an arc of the switching locus,  $W_+^1$ . The solution of  $(l_-)$  through the origin in the second quadrant as a part of the switching locus is called  $W_-^1$ . Starting with an initial point on  $W_+^1$  (or  $W_-^1$ ) and integrating backwards in time until  $\eta_2(t) = 0$  results in a point of the arc  $W_+^2$  (or  $W_-^2$ ) of the switching locus. The initial values on  $W_+^1$  (or  $W_-^1$ ) are  $\eta_2(0) = 0$ ,  $\eta_1(0) = -1$  (or  $\eta_1(0) = +1$ ), the sign is given by corollary 1.4, the magnitude can be chosen at will since the system is homogeneous,  $x(0)$ ,  $y(0)$  are given as coordinates of  $W_+^1$  (or  $W_-^1$ ).  $W_-^2$  (and  $W_+^2$ ) are reflected the same way to obtain  $W_+^3$  (and  $W_-^3$ ), and so on. The switching locus is then the union of all sets  $W_+^k$  and  $W_-^k$ , where  $k$  can be finite or infinite. In general the integration is performed numerically so that the switching locus is determined pointwise.

In the case of attractive force Markus and Lee have shown that  $k > 1$ , if there is no friction,  $F \equiv 0$ ,  $k = \infty$ . In the case of repulsive force  $k = 1$  and the switching locus coincides with the trajectory through the origin in the second and fourth quadrant over its entire length.

In parts of their discussion they assume that  $W(x)$  is single-valued which need not be true.

As an example of a conservative system with attractive force they construct the switching locus for the time-optimal control of a hard spring (Duffing equation). As can be seen from the results the switching locus  $W(x) = \bigcup_{k=1}^{\infty} (W_+^k \cup W_-^k)$  is not single-valued for  $k \geq 3$ .

They also present some interesting results-mainly computer solutions-for the motion of a forced pendulum.

Remark: Later in the analysis singular points are discussed. These are points in the  $(x,y)$  - plane for which  $\dot{x}$  and  $\dot{y}$  are zero simultaneously. For system (1) a singular point can exist only on the  $x$ -axis with  $x$ -coordinate a real root of  $F(x,0) - u = 0$  for  $u = +1$  or  $-1$ . For the conservative case the corresponding equation is  $f(x) - u = 0$ . The reader can easily verify that a singular point  $(\bar{x},0)$  is a vortex if  $f'(\bar{x}) > 0$  and a saddle if  $f'(\bar{x}) < 0$ . A separatrix is a solution trajectory which tends to a saddle point as  $t$  approaches  $+\infty$  or as  $t$  approaches  $-\infty$ .

In the following chapters the author will present results for a system which describes the motion of a soft spring (negative cubic term in the Duffing equation). The domain of controllability will be described and a procedure for obtaining the maximum number of switchings will be presented. Also some peculiar behavior of the switching locus will be discussed and analyzed.

## CHAPTER II

### DOMAIN OF CONTROLLABILITY

#### 2.1 Problem Statement

An initial state of the system

$$\ddot{x} + f(x) = u$$

or

$$\dot{x} = y$$

$$\dot{y} = -f(x) + u, \text{ with } -1 \leq u \leq +1 \quad (1)$$

shall be steered to the origin in minimum time. The synthesis problem is solved by determining the feedback control  $u(x,y)$ . The following conditions are placed on  $f(x)$ :

(i)  $f(x)$  is in the class  $C^1$ .

(ii)  $-f(-x) = f(x)$ .

(iii)  $\frac{df(x)}{dx} = f'(x) > 0$  in some region about the origin, this implies that there exists some  $d > 0$  such that  $f(x) > 0$  for  $0 < x < d$ .

(iv) There exists some  $x_s > 0$  such that  $f(x) < -1$  for  $x > x_s$ .

A possible graph of  $f(x)$  is shown in figure 1. Condition (iv) assures that each system  $(1_+)$  with  $u = +1$  and  $(1_-)$  with  $u = -1$ , as described in section 1.3, has at least one critical point,  $x_{s+}$  and  $x_{s-}$  a saddle. The origin, which is the target for this problem, is a stable equilibrium point, by condition (iii), since there  $u$  is set to zero.

For this investigation the following  $f(x)$  will be considered:

$$f(x) = x - \beta x^3,$$

the characteristic of a soft spring with  $\beta > 0$  as parameter.

System (1) can be integrated in the phase plane by eliminating the time:

$$dt = \frac{dx}{y} \quad (3)$$

Then

$$\frac{dy}{dx} = \frac{u - f(x)}{y} .$$

For  $u = \text{constant}$  one can express the trajectories  $y(x)$  analytically as

$$y(x) = \pm \sqrt{y^2(0) + 2u x - 2 \int_0^x f(\xi) d\xi} \quad (4)$$

With equations (3) and (4) and the specified  $f(x)$ ,  $x(t)$  and  $y(t)$  can be determined analytically. The time needed to steer a point  $(x_c, y(x_c))$  to  $(x, y(x))$  is by equation (3):

$$t = \int_{x_c}^x \frac{dx}{y}$$

with  $f(x) = x - \beta x^3$  and equation (4)

$$t = \int_{x_c}^x \frac{dx}{\sqrt{y^2(0) + 2ux - x^2 + \frac{\beta}{2} x^4}} \quad (5)$$

which is an elliptic integral of the first kind.

The inversion of this elliptic integral [2, pp. 452-455], with  $y^2(x) = h(x)$ , yields

$$x(t) = x_c + \frac{\frac{1}{4} h'(x_c)}{s(t) - \frac{1}{24} h''(x_c)}$$

$$y(t) = - \frac{\frac{1}{4} h'(x_c) \delta(t)}{[c(t) - \frac{1}{24} h''(x_c)]^2},$$

where  $x_c$  is a zero of  $h(x)$ ;  $h'(x) = \frac{dh(x)}{dx}$ .  $\delta(t)$  is the Weierstrass function. Its definition and characteristics are presented in [2, Chapter 3],  $\delta(t) = \frac{d\delta(t)}{dt}$ .

## 2.2 Domain of Controllability, Existence of an Optimal Control

As mentioned in section 1.4 in this section the domain of controllability for system (1) with conditions (i)....(iv) on  $f(x)$  will be described, and the existence of an optimal control will be proved.

Theorem 2.2 Given the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -f(x) + u\end{aligned}\tag{1}$$

where  $u$  is measurable on  $-1 \leq u \leq +1$  with  $f(x)$  as described in section 2.1 satisfying conditions (i)...(iv). The domain  $G$  of controllability is the open connected band  $B$  between the principal separatrices through  $x_{s+}$  and  $x_{s-}$ . Furthermore, each point in  $B$  can be steered to the origin by an optimal control  $u(t) \in \Delta$ .

The principal separatrices are the trajectories through  $x_{s+}$  and  $x_{s-}$  indicated as solid lines in the phase plane, figure 1.

PROOF: By theorem 1.1 there exists a neighborhood of the origin which lies in  $G$ . Thus, one must show that every point in  $B$  can be steered into this neighborhood. Start at point  $P$  (compare figure 1). Choosing  $u = 0$  the trajectory follows a solution of

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -f(x)\end{aligned}$$

Then  $y$  will increase and  $x$  will decrease. At  $x < d, f(x) > 0$ , therefore, to continue in the same direction set  $u = +1$ . If  $f(x) > 1$  for some value of  $x < d$  (see dashed line for  $f(x)$  in figure 1)  $y$  will decrease. But the trajectory cannot leave the region  $P$  bounded by the principal separatrix through  $x_{f+}$ , that is for a maximal constant  $u = +1$ . If the trajectory tends to intersect the positive  $x$ -axis set  $u = 0$  and  $y$  will decrease, while  $x$  is still decreasing. Thus, one can steer the trajectory into the third quadrant with a properly chosen  $u$ . In the third quadrant any value  $0 \leq u \leq 1$  will steer the trajectory towards the negative  $x$ -axis. As it enters the second quadrant  $x$  and  $y$  will increase. To steer it toward the origin  $u = -1$  is needed. As it encircles the origin with decreasing distance from the origin one needs only to pick the proper  $u$  to continue in the wanted direction.

If  $P$  lies in one of the other quadrants the proof is similar.

The reader should also observe that a trajectory which enters the region  $Q$ , bounded by the four separatrices closest to the origin (see curvilinear quadrilateral inside the solid and dashed separatrices in figure 1) will not leave it again.

Thus, it has been shown that each initial point  $P \in B$  can be steered to the origin by some control  $u(t) \in A$ . Since  $x$  and  $y$  remain bounded during this process by theorem 1.2 there exists an optimal control. This completes the proof.

The proof is basically the same for the case when positive damping is present.

By the maximum principle and the Theorem of Interlacing Zeros the optimal controller is a relay controller:  $u$  assumes the values  $-1$  and  $+1$ .

### 2.3 Phase Plane Analysis

Since the behavior of the trajectory through the origin is important for the switching locus a detailed discussion of the relation between  $f(x)$  and the trajectory through the origin is presented.

For this trajectory in equation (4)  $y(0) = 0$  and

$$y(x) = \pm \sqrt{2 u x - 2 \int_0^x f(\xi) d\xi}$$

With the function as described in section 2.1 one must distinguish three different cases.

- (a) Each system  $(1_+)$  with  $u = +1$  and  $(1_-)$  with  $u = -1$  has one singular point as the only real solution of each equation

$$f(x) = 1 - x_{s+}$$

$$\text{and } f(x) = -1 - x_{s-}; \quad (4)$$

see also the graph in figure 2. The trajectory through the origin (solid line for  $u = +1$ , dashed line for  $u = -1$ ) has negative slope throughout the second and fourth quadrants.

This case applies to the above example for  $\beta > \frac{4}{27}$ .

- (b) Each system has three singular points, i.e., each equation of (4) has three real roots. The graph of  $f(x)$  and the phase plane are shown in figure 3. For  $u = +1$  the separatrices and trajectory are sketched in solid lines, for  $u = -1$  they are indicated in dashed lines. The smallest root,  $x_o$ , is a vortex, whereas  $x_t$  and  $x_s$  represent saddles. The separatrix through  $x_t$  encircles  $x_o$ . Its reintersection with the  $x$ -axis lies in the same half plane as  $x_t$ . Therefore, the trajectory through the origin does not reintersect the  $x$ -axis. However it changes its slope. The slope is negative on  $(-\infty, x_{t-})$ ,

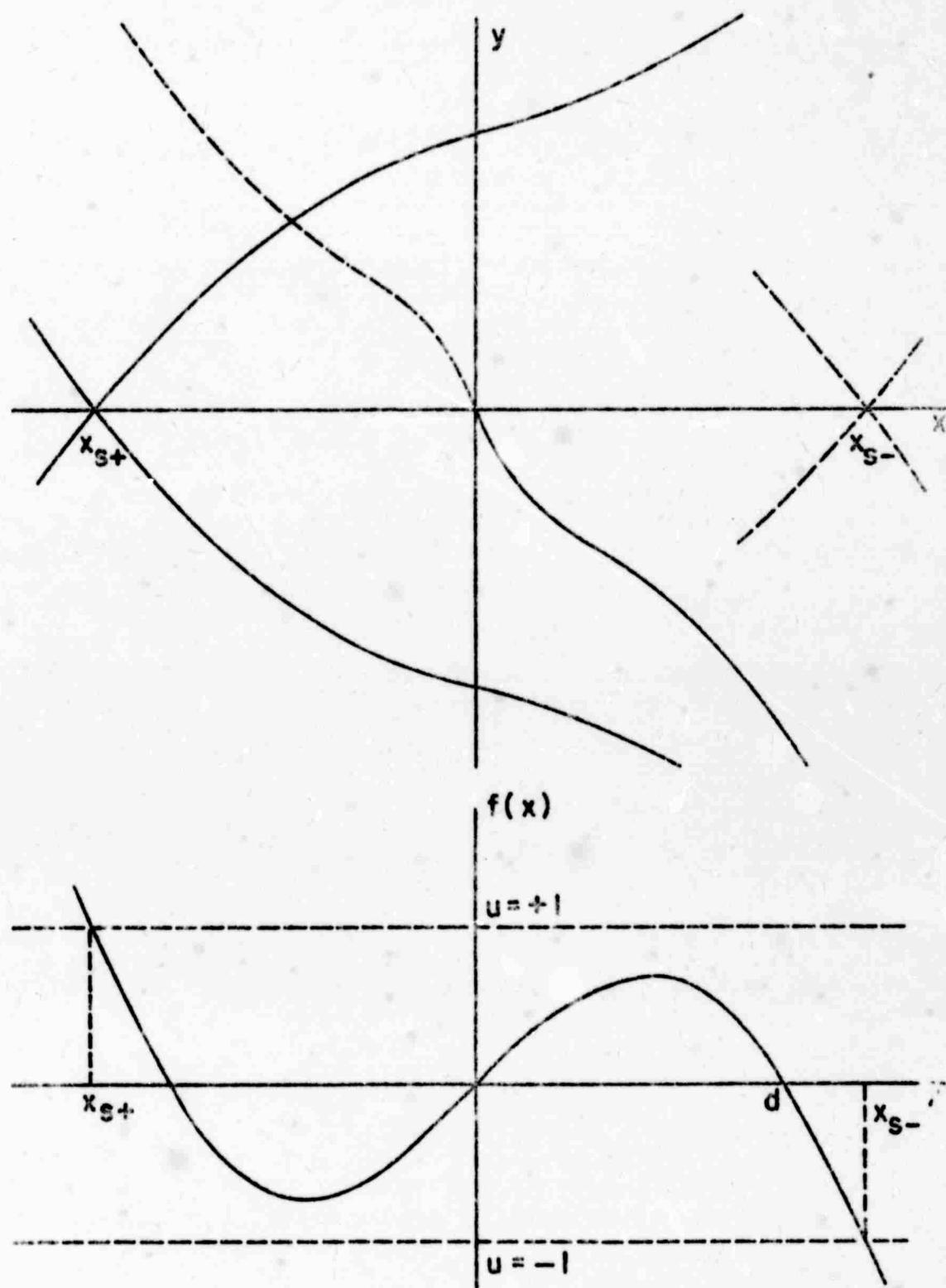


FIGURE 2. ILLUSTRATION OF CASE (a).

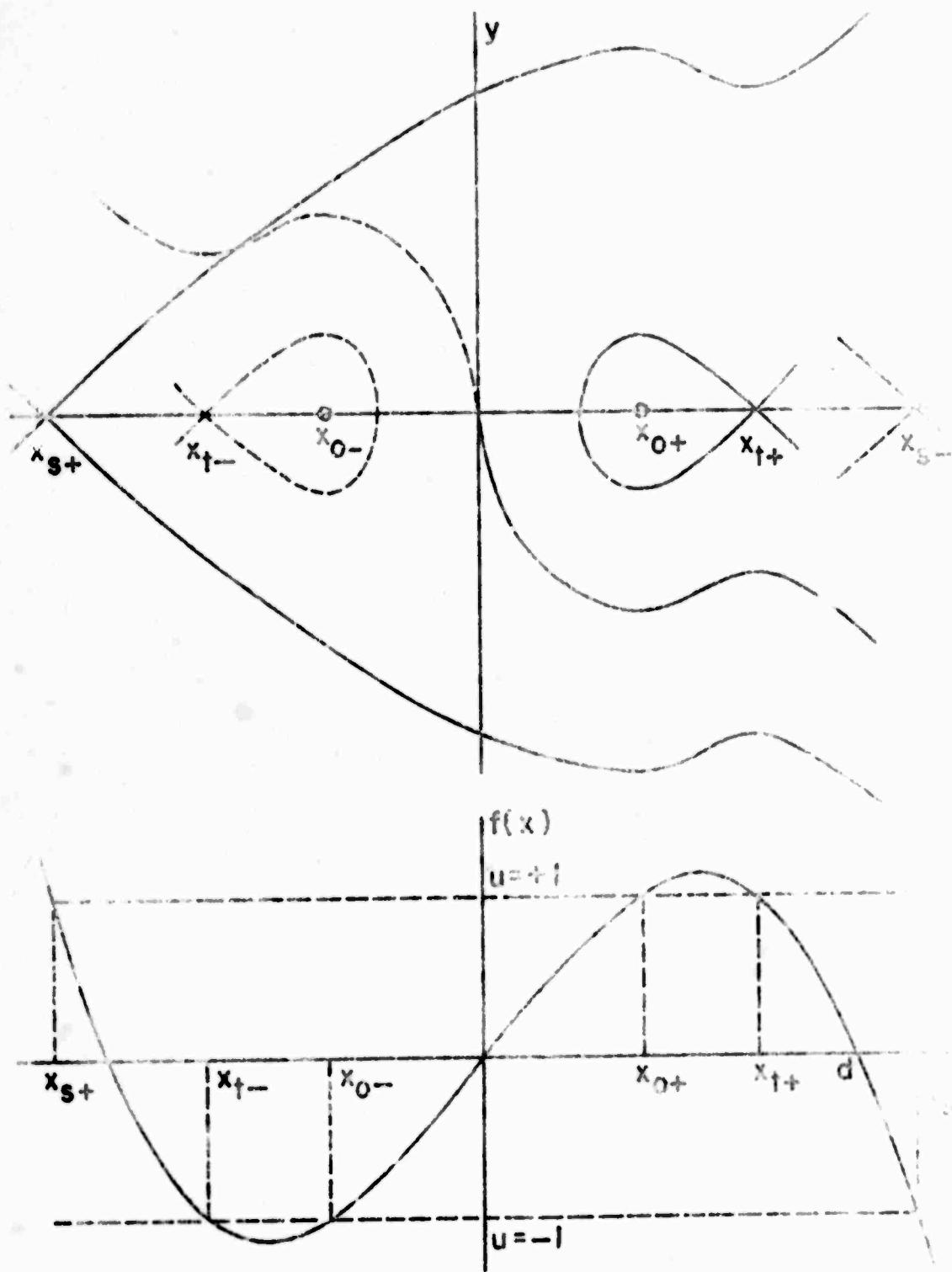


FIGURE 3. ILLUSTRATION OF CASE (b).

$(x_{0-}, x_{0+})$ , and  $(x_{t+}, \infty)$  and positive on  $(x_{t-}, x_{0-})$  and  $(x_{0+}, x_{t+})$ ,  
at  $x_{0\pm}$  and  $x_{t\pm}$  the slope is zero.

This holds for the example when  $\frac{2}{27} < \beta < \frac{4}{27}$ . The limiting case between cases (a) and (b) is the case when  $f(x)$  just touches the lines  $u = +1$  and  $u = -1$ . Each equation (4) has three real roots, a simple root  $x_s$  and a double root  $x_0 = x_t$ . When  $x$  has the value of this double root the trajectory through the origin has zero slope, except at these two points it has negative slope throughout the second and fourth quadrants. For the example  $\beta = \frac{4}{27}$ .

(c) Each system has three singular points as in (b). But the separatrix through  $x_t$  reintersects the  $x$ -axis in the other half plane, see figure 4. So the trajectory through the origin lies inside the region surrounded by the separatrix through  $x_t$  and therefore reintersects the  $x$ -axis. This holds for  $\beta < \frac{2}{27}$ .

The limiting case between cases (b) and (c) is the case when the separatrix through  $x_t$  reintersects the  $x$ -axis in the origin, that is, the trajectory through the origin coincides with this separatrix and reintersects the  $x$ -axis in a singular point; this occurs for  $\beta = \frac{2}{27}$ .

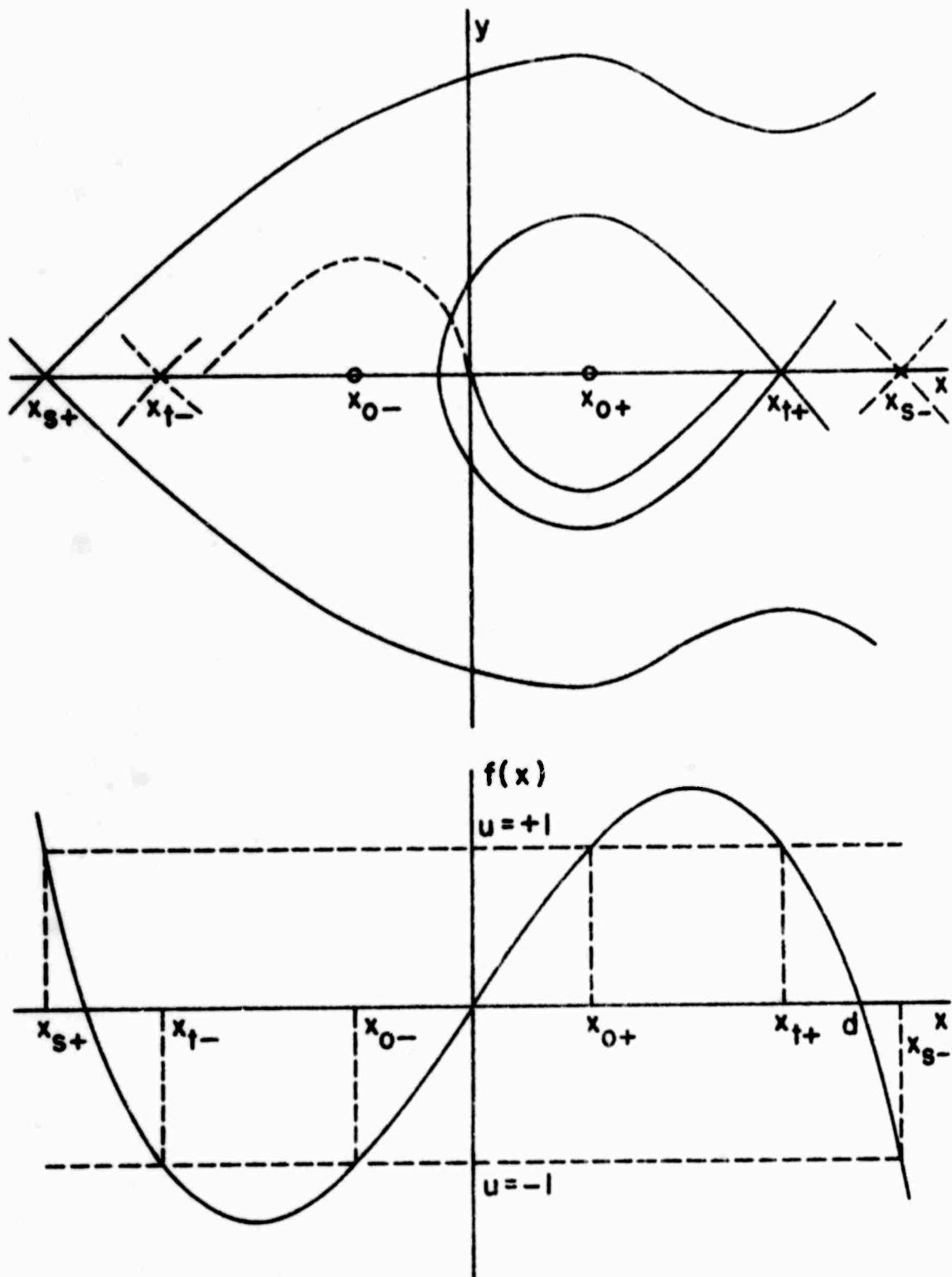


FIGURE 4. ILLUSTRATION OF CASE (c).

### CHAPTER III

#### SWITCHING LOCUS FOR SMALL VALUES OF $\beta$

In this chapter the author will discuss the case (c), i.e., for the example only values  $\beta < \frac{2}{27}$  will be considered. Then for  $0 \leq \beta < \frac{2}{27}$  one can find

$$1 \leq x_0 < 1.1$$

Once  $x_0$  is determined one obtains

$$x_s = -\sqrt{\frac{1}{\beta} - \frac{3}{4} x_0^2} - \frac{x_0}{2}$$

$$x_t = \sqrt{\frac{1}{\beta} - \frac{3}{4} x_0^2} - \frac{x_0}{2}$$

For small  $\beta$  one can approximate

$$x_0 = 1, x_s = -\frac{1}{\sqrt{\beta}} - \frac{1}{2}, \quad x_t = \frac{1}{\sqrt{\beta}} - \frac{1}{2}.$$

#### 3.1 Switching Locus-Maximum Number of Switchings

For the given system (1) the theorem of interlacing zeros which was stated in section 1.4 as basic for the construction of the switching locus is equivalent to Sturm's comparison theorem for differential equations [4]. For one can write the second-order differential equations for  $y$  and  $\eta_2$ :

$$\ddot{y} + f'(x) y = 0$$

$$\text{and} \quad \ddot{\eta}_2 + f'(x) \eta_2 = 0.$$

As stated in section 1.4 the segments of the trajectory through the origin in the second and fourth quadrants belong to the switching locus, which are  $W_-^1$  and  $W_+^2$ . One follows the procedure described in 1.4 to obtain consecutive arcs of  $W_\pm^k$ . Burmeister presents a different procedure for the construction of the switching locus in [5], for which he applies the variation equations of the system. He claims that his method is quicker and more accurate since fewer integrations are performed. The author compared Burmeister's method with the one described in 1.4 for  $\ddot{x} + x + 2x^3 = u$ . The former yielded only two arcs of the switching locus of desirable accuracy and took relatively more computer time than the latter which yielded four arcs of desirable accuracy.

If a switching lies on the x-axis (a zero of  $\eta_2$  is a zero of y) then the next switching must also lie on the x-axis. Thus, consider the arc  $W_+^1$  in the fourth quadrant. It reintersects the positive x-axis in  $x_{1+}$ , figure 5. Starting at this point backwards in time with  $u = -1$  the trajectory will reintersect the negative x-axis in  $x_{2-}$  which is also the endpoint of  $W_-^2$ . The trajectory leaving  $x_{2-}$  with  $u = +1$  will reintersect the positive x-axis in  $x_{3+}$ , the endpoint of  $W_+^3$ , provided the trajectory lies still inside the region bounded by the separatrix through  $x_t$  (see dashed line in figure 5). If it lies outside this region (in figure 5 the trajectories leaving  $x_{3-}$  and  $x_{3+}$ ) it will stay between the principal separatrices and go to infinity in G.

This shows a way of determining the maximum possible number of switchings, namely, follow the trajectories whose switching points lie on the x-axis until they leave towards infinity and count the intersections  $x_i$  on the positive or negative x-axis. This number of intersections

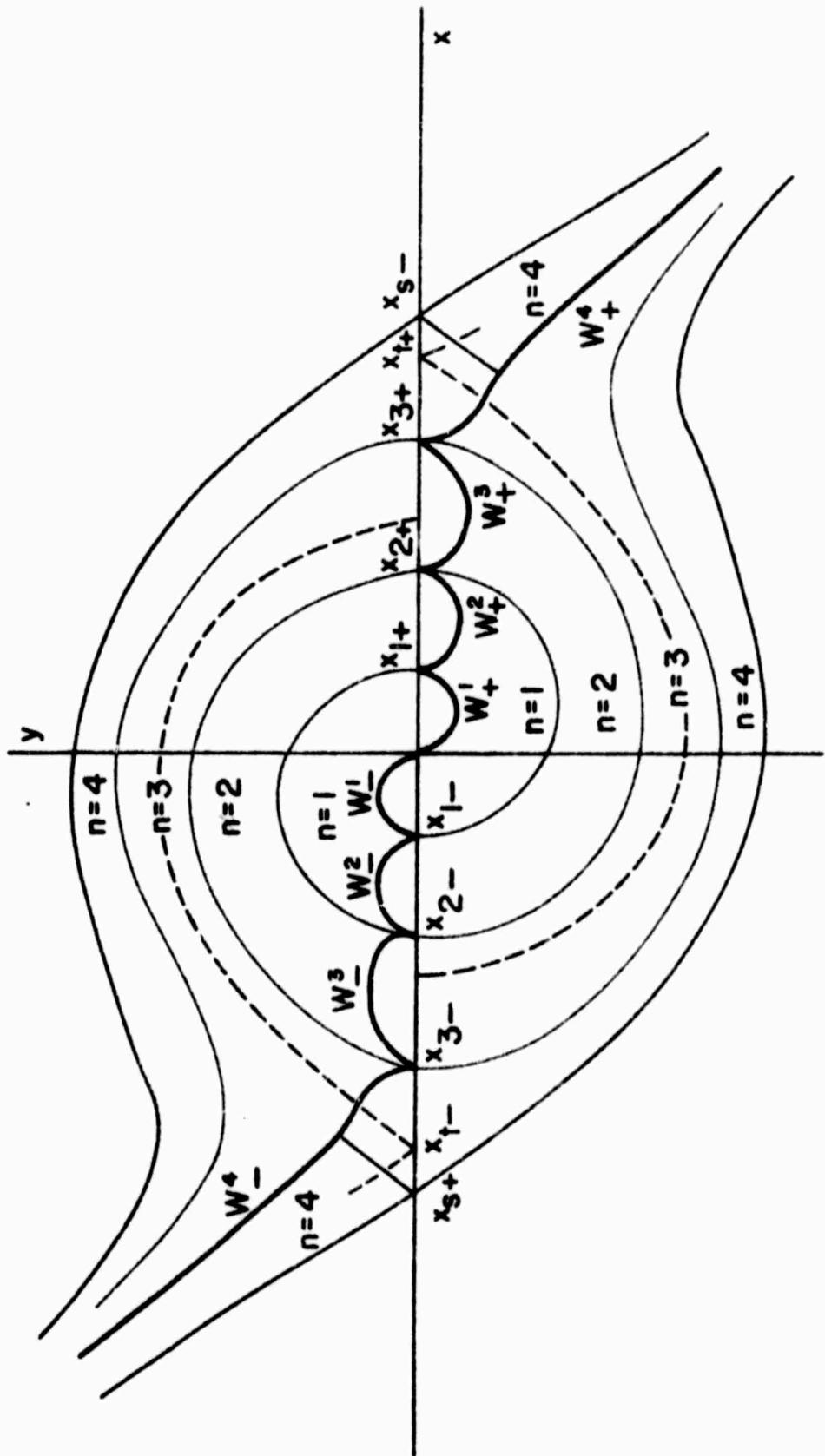


FIGURE 5. SUBREGIONS OF  $G$  WITH THE CHARACTERISTIC NUMBERS OF SWITCHINGS.

(including the origin) is then the maximum number of switchings. For the above example  $f(x) = x - \beta x^3$  with  $\beta < \frac{2}{27}$  the following algorithm has been developed:

The trajectory through the origin for  $u = +1$ ,  $w_+^1$ , is

$$y(x) = -\sqrt{\frac{\beta}{2}x^4 - x^2 + 2x}$$

Its reintersection with the x-axis,  $x_{1+} = x_1$  is determined by solving

$$\frac{\beta}{2}x^4 - x^2 + 2x = 0,$$

that is finding the smallest positive root of

$$\frac{\beta}{2}x^3 - x + 2 = 0.$$

Note that by the symmetry of  $f(x)$   $x_{1-} = -x_1$ . Then the next step is to determine the intersection  $x_2$  for the trajectory leaving  $-x_1$ .

This trajectory is  $y(x) = -\sqrt{\frac{\beta}{2}x^4 - x^2 + 2x + 4x_1}$ , and  $x_2$  is determined by solving

$$\frac{\beta}{2}x^4 - x^2 + 2x + 4x_1 = 0$$

for the smallest positive root. Then the n-th reintersection is found with the knowledge of the preceding intersections by solving

$$\frac{\beta}{2}x^4 - x^2 + 2x + 4 \sum_{i=1}^{n-1} x_i = 0.$$

Since  $x_{n-1}$  is known and  $-x_{n-1}$  is also a root of the above equation this fourth degree equation can be reduced to a cubic equation

$$x^3 - x_{n-1}x^2 - \left(\frac{2}{\beta} - x_{n-1}\right)x + \left(\frac{4}{\beta} + \frac{2}{\beta}x_{n-1} - x_{n-1}^3\right) = 0,$$

which can be solved exactly;  $x_n$  is the smallest positive root. However, for computer solution the above fourth degree polynomial can be solved more quickly by Newton's method (or any other algorithm) with a starting

value  $x_n = x_{n-1} + 2$  than the cubic equation which involves variable transformations. And one also must determine the smallest root of the three roots.

The above procedure is terminated when  $x_n$  approaches  $x_t$ , when the trajectory leaving  $-x_{n-1}$  reintersects the positive  $x$ -axis between the reintersection of the separatrix through  $x_{t-}$  ( $u = -1$ ) and  $x_{t+}$ . For the trajectory leaving the corresponding point  $-x_n$  the polynomial has no positive root  $x < x_t$ .

These trajectories with the switching points on the  $x$ -axis divide the domain  $G$  of controllability into subregions for each of which a certain number of switchings holds, e.g., in figure 5 for each point in a region marked with  $n = 4$  four switchings are necessary to steer it to the origin in minimum time. The outer boundary of each region belongs to it, except the boundary of  $G$ , the separatrices through  $x_{s+}$  and  $x_{s-}$ , which does not belong to  $G$ . The switching at the origin to  $u = 0$  is not included. The example shown in figure 5 is plotted for  $\beta = 0.01$ .

For the linear case,  $\beta = 0$ , the harmonic oscillator, the difference between two succeeding intersections of the switching locus is

$x_i - x_{i-1} = 2$  and  $x_n = 2n$ . For  $\beta > 0$  it is  $x_i - x_{i-1} = 2 + \delta_i$  with  $\delta_i$  increasing for increasing  $i$ . In Markus-Lee's example which corresponds to  $\beta < 0$  the difference  $x_i - x_{i-1}$  is decreasing for increasing  $i$ . As a rough estimate for the maximum number of switchings for  $0 < \beta < \frac{2}{27}$  one may assume the linear case and check how many intersections with  $x_i - x_{i-1} = 2$  are possible on  $(0, x_t)$  or  $n < \frac{x_t}{2}$  (for very small  $\beta$   $n < \frac{1}{2} (\frac{1}{\beta} - \frac{1}{2})$ ).

The switching locus of an example with  $\beta = 0.03$  is shown in figure 6. Here the maximum number of switchings is two.

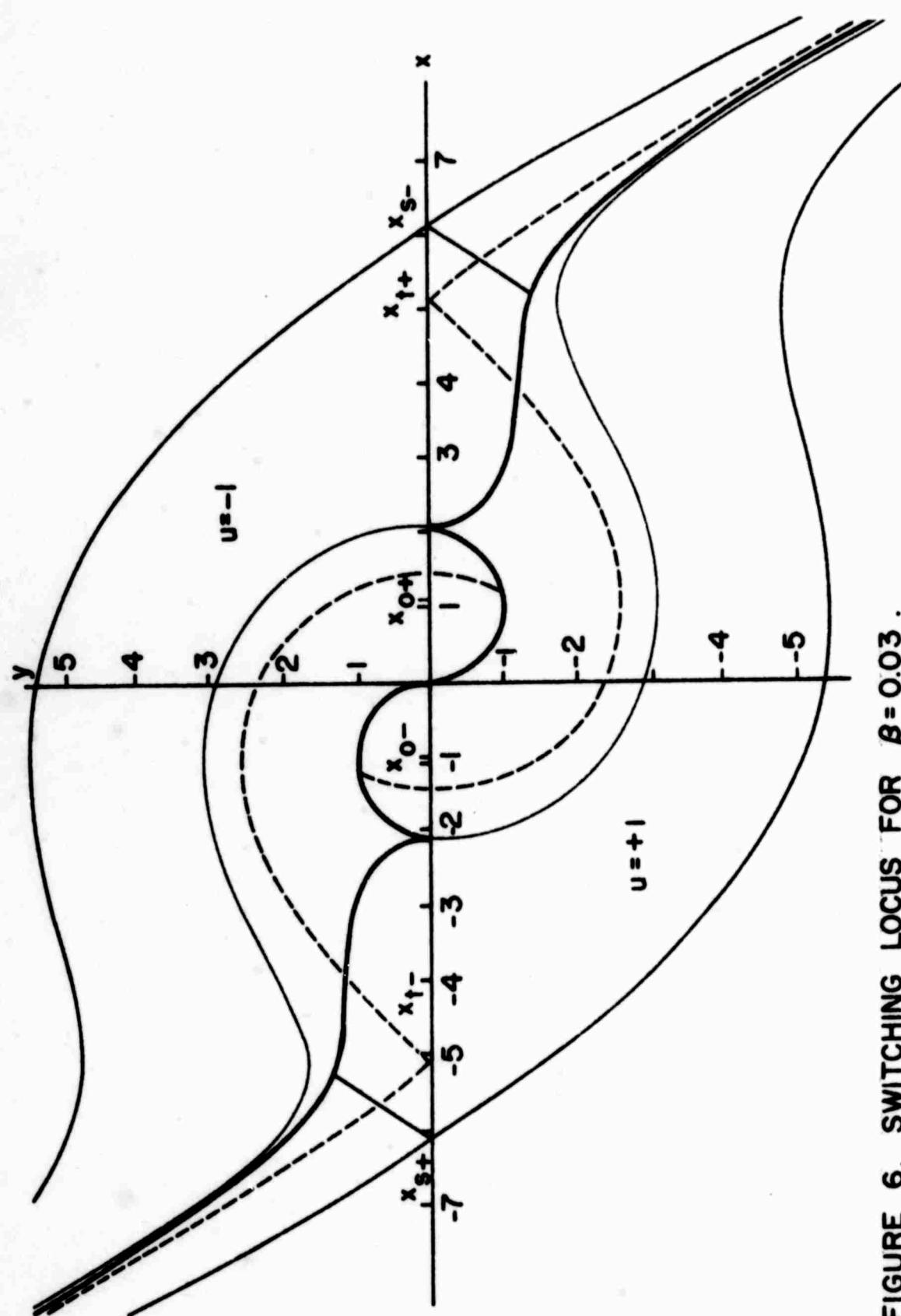


FIGURE 6. SWITCHING LOCUS FOR  $\beta = 0.03$ .

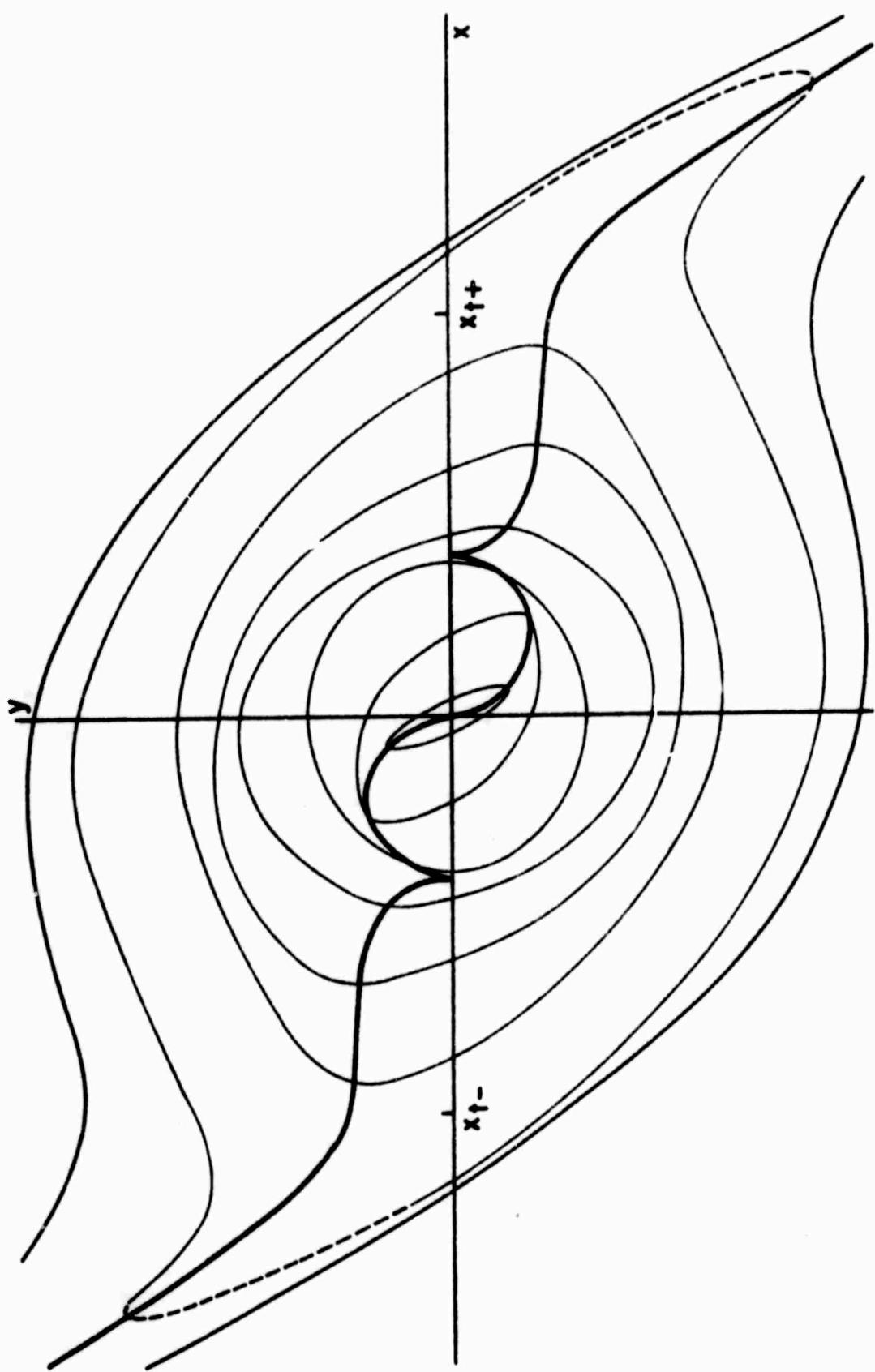


FIGURE 7. ISOCRHONES FOR  $\beta = 0.03$ .

In figure 7 the isochrones are shown for the preceding example ( $\beta = 0.03$ ). Their equation is

$$T(x,y) = \int_0^x \frac{dy}{y(0)} = \text{constant, for the time-optimal } u.$$

Each point on such a closed curve is steered to the origin time-optimally in the same length of time.

These isochrones represent also boundaries for the sets of attainability with the origin as initial point, e.g., the isochrone with  $T(x,y) = T_k$  surrounds a region each point of which can be reached from the origin with the time optimal control in the time  $t \leq T_k$ . From this illustration and other examples one may conjecture that these regions are convex if they lie within  $x_{t_-} < x < x_{t_+}$ .

### 3.2 Influence of Damping

The presence of damping affects the behavior of the system significantly. Since the nonlinear system with damping cannot be analyzed the linear system will be considered first. The differential equation satisfied by a time-optimal control is given by  $\ddot{x} + \alpha \dot{x} + x = \text{sgn } \eta_2$ . If  $\dot{x} = y$  then,  $y' + \alpha y + y = 0$ , and the equation satisfied by  $\eta_2$  is

$$\ddot{\eta}_2 - \alpha \dot{\eta}_2 + \eta_2 = 0$$

The system oscillates only for  $\alpha < 2$  ( $\alpha = 0$  represents the harmonic oscillator) so for  $\alpha > 2$  the maximum number of switchings is one, because

$$\eta_2(t) = \frac{e^{\frac{\alpha}{2} t}}{\sqrt{\frac{\alpha^2}{4} - 1}} \sinh \sqrt{\frac{\alpha^2}{4} - 1}$$

backwards in time, with the only zero at  $t = 0$ . For  $\alpha = 2$

$$\eta_2(t) = (At + B) e^{\frac{\alpha}{2} t}, \text{ where}$$

A and B are positive constants. Therefore the maximum number of switchings is one. For  $\alpha < 2$  with

$$\eta_2(t) = \frac{\frac{\alpha}{2}t}{\sqrt{1 - \frac{\alpha^2}{4}}} \sin \sqrt{1 - \frac{\alpha^2}{4}} t$$

the system has an infinite maximum number of switchings in  $G = \mathbb{R}^2$ . The trajectories with switchings on the x-axis are described by

$$x = 1 + \frac{A - \frac{\alpha}{2}t}{\omega} e^{\frac{\alpha}{2}t} \cos(\omega t + \arcsin \frac{\alpha}{2})$$

$$y = \frac{A - \frac{\alpha}{2}t}{\omega} e^{\frac{\alpha}{2}t} \sin \omega t.$$

The switching intervals are  $\omega t = 0, \pi, 2\pi, \dots, \omega = \sqrt{1 - \frac{\alpha^2}{4}}$ . Considering the trajectory for  $u = +1$  leaving the x-axis at  $-x_n$  one obtains

$$x_n = 1 + (1 + x_{n-1}) e^\rho$$

with  $\rho = \frac{\frac{\alpha}{2}\pi}{\sqrt{1 - \frac{\alpha^2}{4}}}$ , (A similar result appears in [6] with a

presentation of the equation for the switching locus). In this linear case one can determine  $x_n$  without first calculating  $x_{n-1}$

$$x_n = 1 + e^{np} + \frac{e^{np} - \rho}{e^\rho - 1}.$$

As the damping increases the distance between these intersections will increase.

Now assume some maximum value  $x \leq x_N$  is given and one wants to know how many switchings of the time-optimal control are needed to steer this point to the origin. One replaces in the above equation  $x_n$  by  $x_N$  and solves for n:

$$n \leq \frac{1}{\rho} \ln \frac{(x_n - 1)(e^\rho - 1) + 2\rho}{e^\rho + 1}.$$

This number may be changed by 1, depending on whether the point with  $x_n$  as abscissa lies above or below the switching locus.

With the nonlinear term the equations for  $y$  and  $\eta_2$  are

$$\ddot{y} + \alpha \dot{y} + f'(x) y = 0$$

$$\ddot{\eta}_2 - \alpha \dot{\eta}_2 + f'(x) \eta_2 = 0$$

with  $f'(x) = 1 - 3\beta x^2$

The solution oscillates for  $f'(x) - \frac{\alpha^2}{4} > 0$  or for

$$x < \sqrt{\frac{1}{3\beta} \left(1 - \frac{\alpha^2}{4}\right)} . *$$

A rough estimate on the maximum number of switchings can be obtained by substituting

$$x_N = \sqrt{\frac{1}{3\beta} \left(1 - \frac{\alpha^2}{4}\right)}$$

in the above inequality for  $n$ .

In figures 8, 9, and 10 an example for  $\beta = 0.005$  is shown to demonstrate how the damping changes the shape of the switching locus and the maximum number of switchings.

---

With  $Z = e^{\frac{\alpha^2}{2} t}$  one obtains for the above equation  $\ddot{Z} + (f'(x(t)) - \frac{\alpha^2}{4})Z = 0$  for which the comparison theorem [4] can be applied.

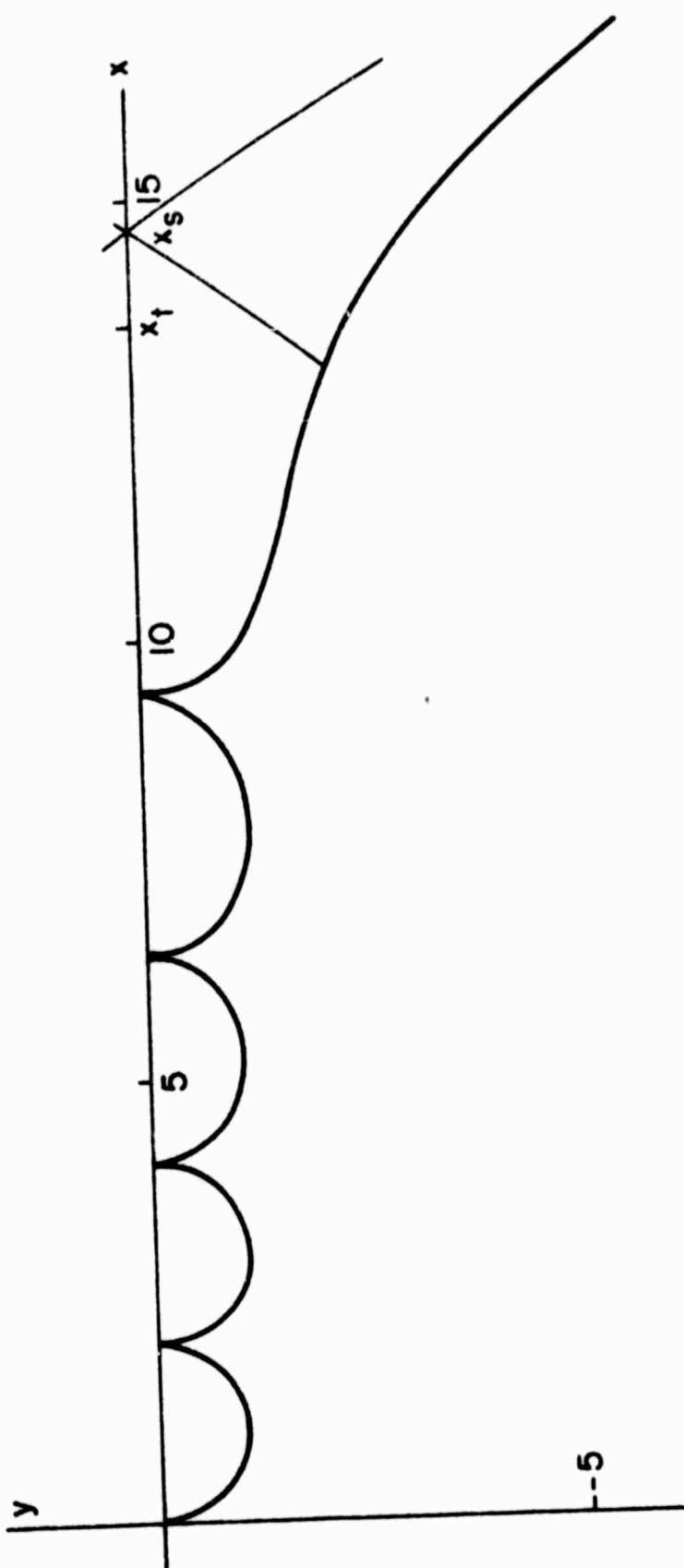


FIGURE 8. SWITCHING LOCUS FOR  $\beta=0.005$ , NO DAMPING.

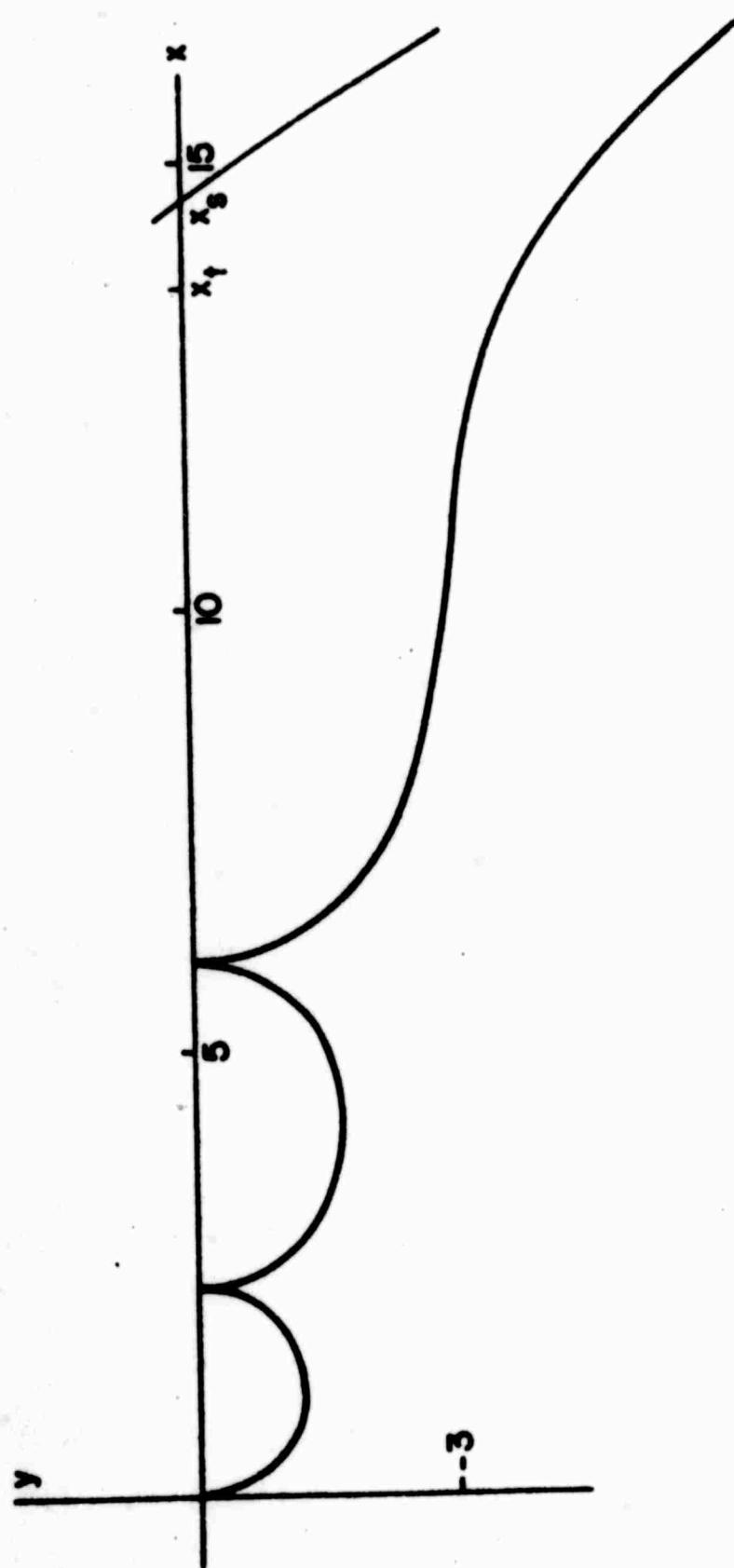


FIGURE 9. SWITCHING LOCUS FOR  $\beta = 0.005$ , WITH DAMPING  $\alpha = 0.2$ .

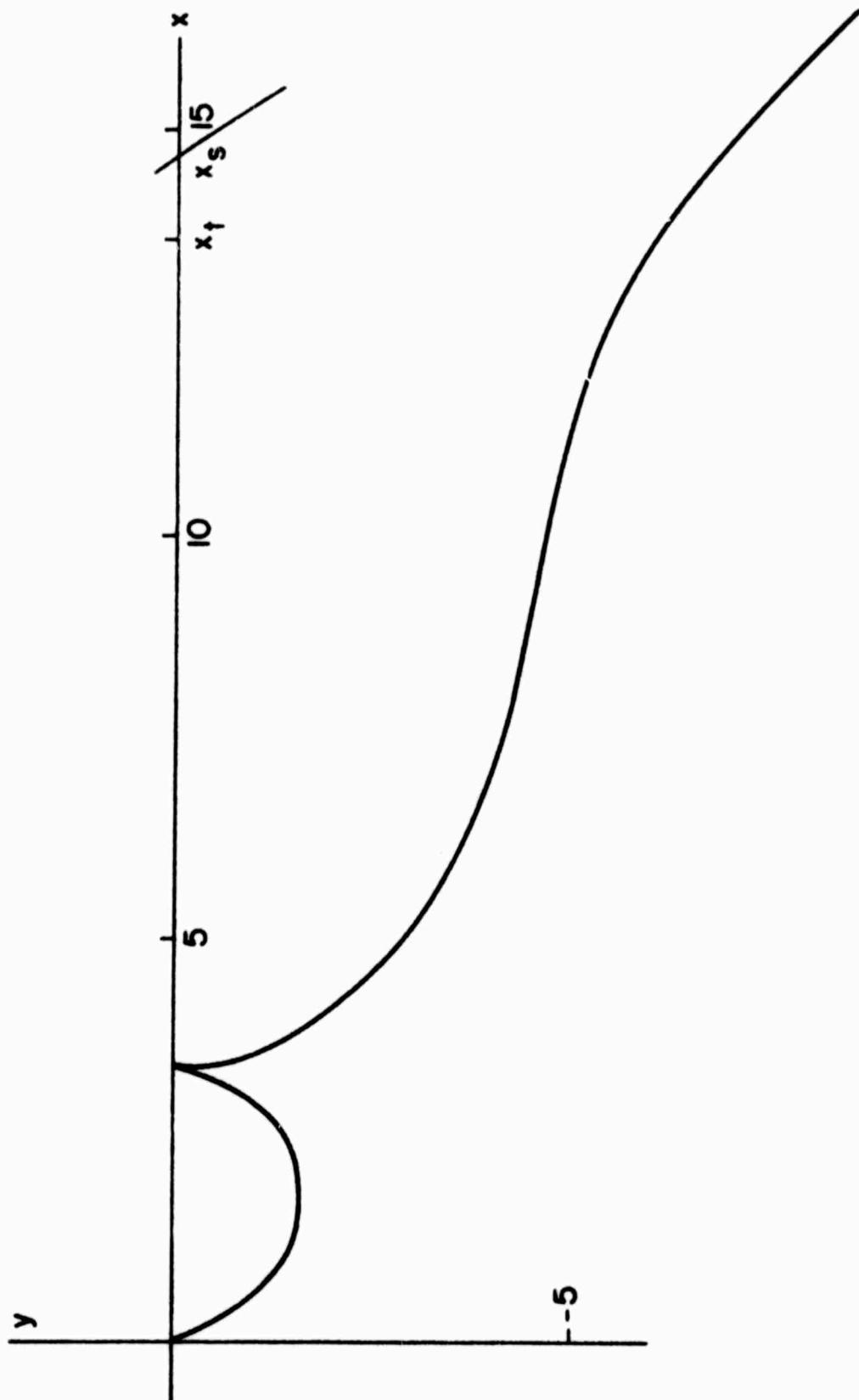


FIGURE 10. SWITCHING LOCUS FOR  $\beta=0.005$ , WITH DAMPING  $\alpha=0.5$ .

CHAPTER IV  
SWITCHING LOCUS FOR INTERMEDIATE  
AND LARGE VALUES OF  $\beta$

In this chapter the cases (a) and (b) as described in section 2.3 will be discussed. Originally the assumption was made that if the trajectory through the origin does not reintersect the x-axis then it coincides with the switching locus over its entire length, and the maximum number of switchings is one. But computer results showed that there are cases for  $\beta > \frac{2}{27}$  for which there exists an arc of the switching locus sticking back from infinity. Furthermore the switching locus is discontinuous, the arc through the origin is not connected with the one sticking back from infinity. Figures 11, 12 and 13 show examples. Two different integration procedures were applied to assume the result. One integration was with respect to time  $t$ , the other procedure was with respect to  $x$  as discussed later. From these results the conclusion was drawn that there exist arcs of the switching locus sticking back from infinity only for some subinterval of  $\frac{2}{27} < \beta < \frac{4}{27}$ . In the following sections the author will consider certain situations for  $\beta > \frac{2}{27}$  and show whether there exists a second switching or not.

#### 4.1 Derivations

The system equations are

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -f(x) + u\end{aligned}\tag{1}$$

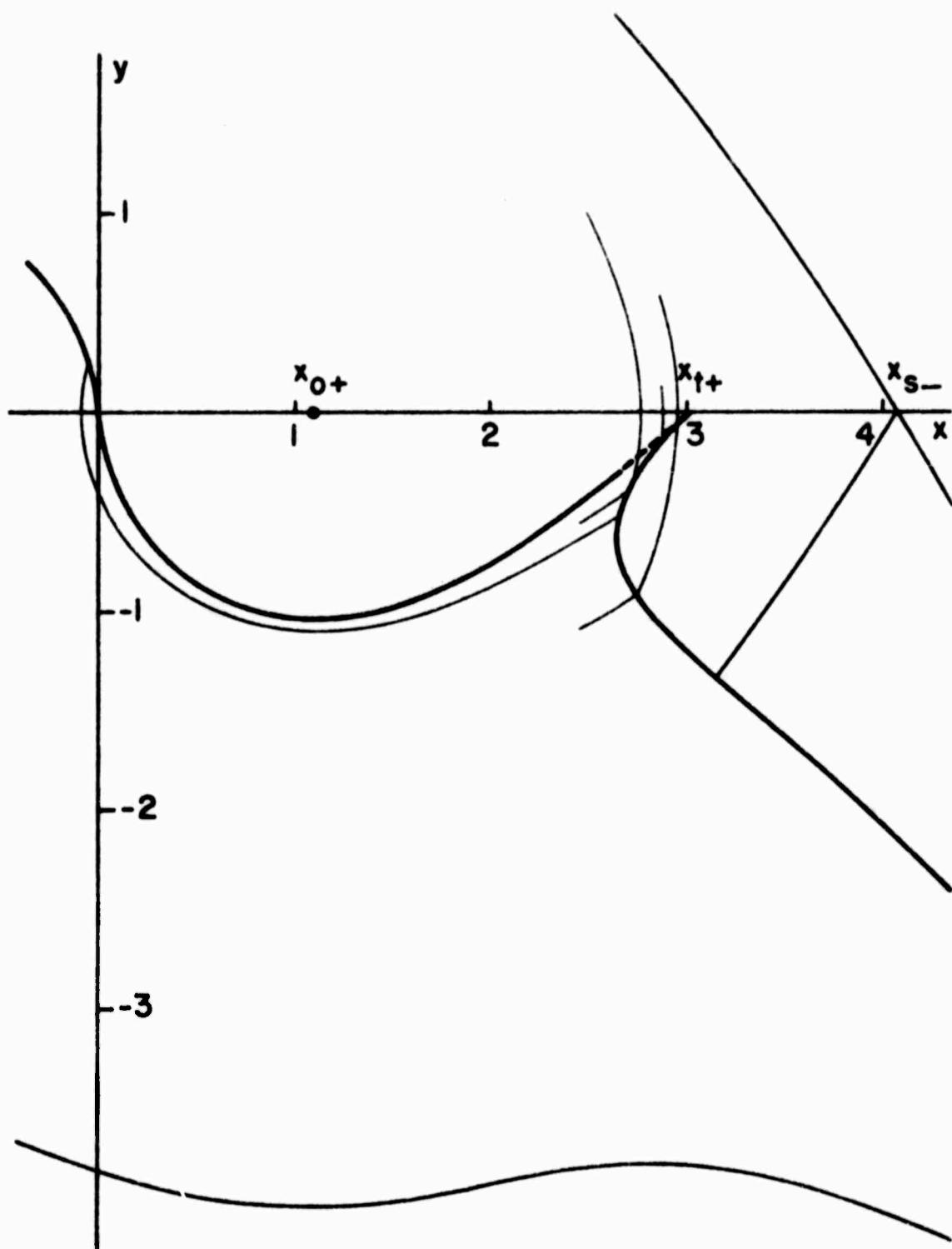


FIGURE II. SWITCHING LOCUS FOR  $\beta = 2/27$ .

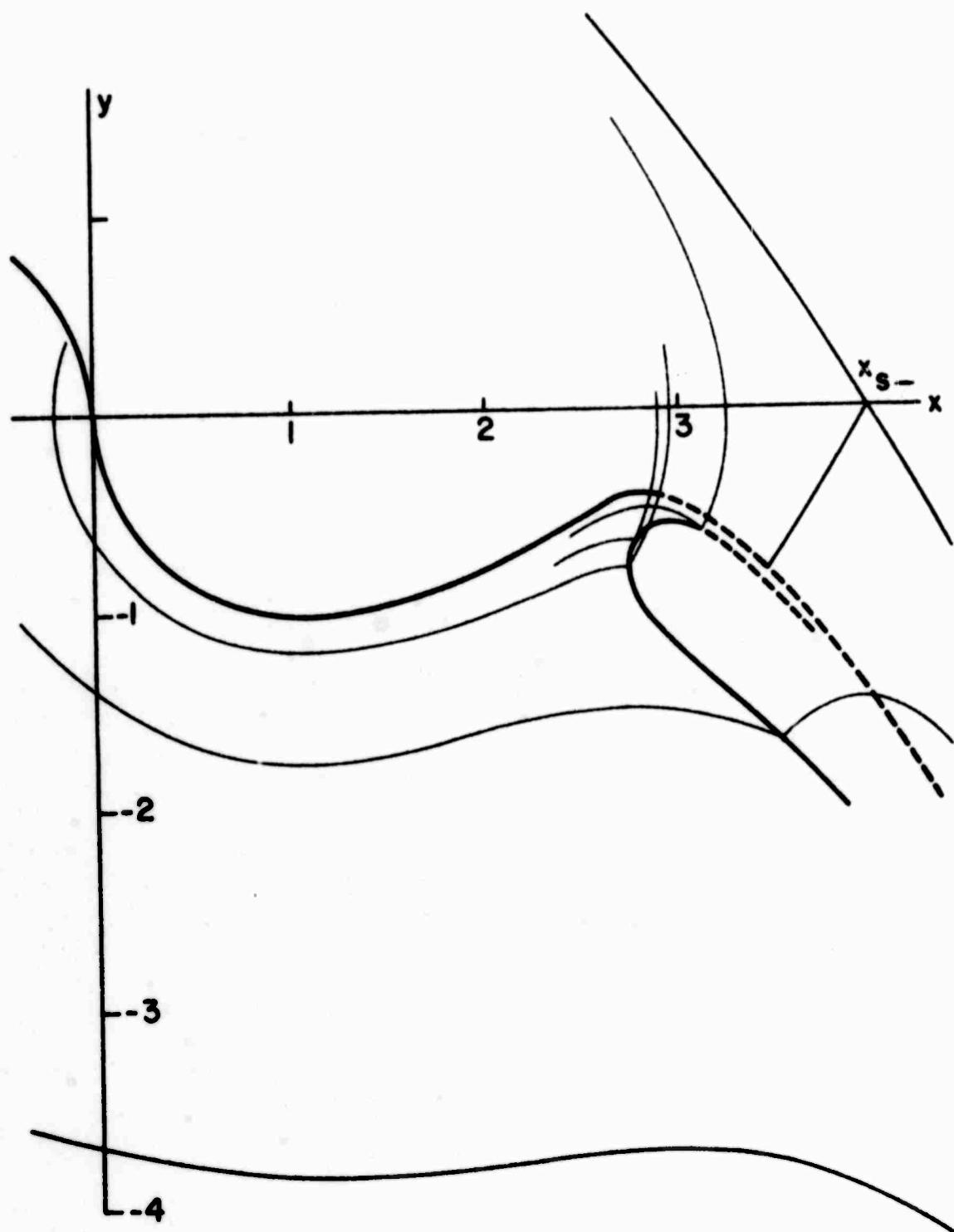


FIGURE 12. SWITCHING LOCUS FOR  $\beta = 0.08$ .

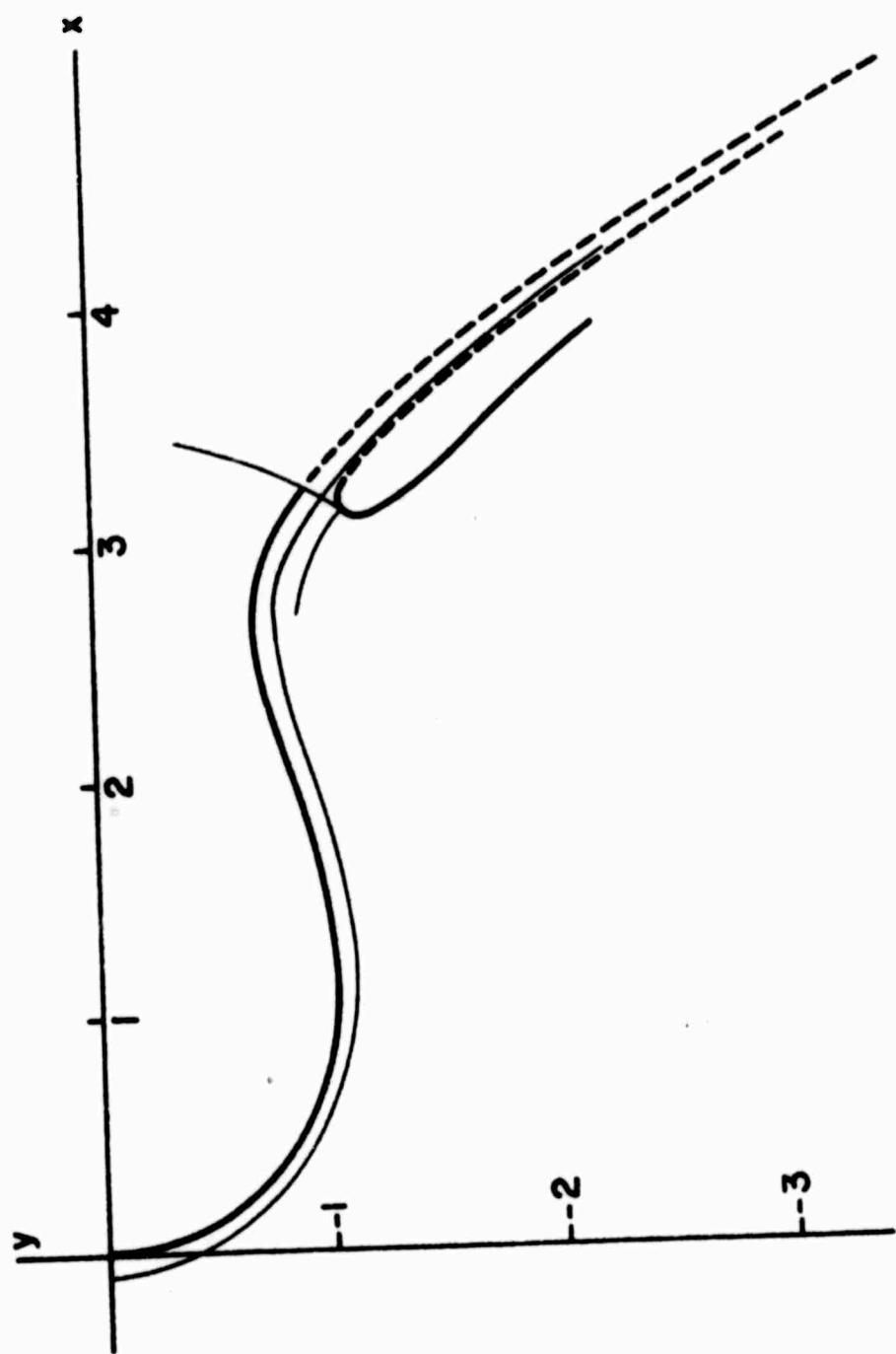


FIGURE 13. SWITCHING LOCUS FOR  $\beta=0.09$ .

with  $u = +1$  below and  $u = -1$  above the switching locus. The adjoint system is

$$\begin{aligned}\dot{\eta}_1 &= f'(x) \eta_2 \\ \dot{\eta}_2 &= -\eta_1\end{aligned}\tag{2}$$

and the Hamiltonian

$$H(x, y, \eta_1, \eta_2) = \eta_1 y + \eta_2 [u - f(x)] = h, \tag{6}$$

where  $h$  is a positive constant along an optimal trajectory. The equation of an optimal trajectory was derived in section 2.1 as

$$y(x) = \pm \sqrt{y^2(0) + 2u x - 2 \int_0^x f(\xi) d\xi}. \tag{4}$$

The same procedure is applied to obtain  $\eta_1$  and  $\eta_2$  as functions of  $x$  and  $y$

$$\begin{aligned}\frac{d\eta_1}{dx} &= \eta'_1 = \frac{f'(x)}{y} \eta_2 \\ \frac{d\eta_2}{dx} &= \eta'_2 = -\frac{\eta_1}{y}.\end{aligned}\tag{2'}$$

One uses the Hamiltonian, equation (6), to eliminate  $\eta_1$ :

$$\eta_1 = \frac{h}{y} - \frac{\eta_2}{y} [u - f(x)].$$

With  $u - f(x) = y y'$  from system (1) one obtains a first-order differential equation for  $\eta_2$  in  $x$ .

$$y \eta'_2 - y' \eta_2 = -\frac{h}{y}$$

or

$$\left(\frac{\eta_2}{y}\right)' = -\frac{h}{y^3}$$

which can be integrated

$$\frac{\eta_2(x)}{y(x)} = \frac{\eta_2(x_0)}{y(x_0)} - h \int_0^x \frac{d\xi}{y^3(\xi)}, \quad y \neq 0. \quad (7)$$

Thus, one can write the feedback control in analytic form:

$$u(x, y) = \operatorname{sgn} \left\{ y(x) \left[ \frac{\eta_2(x_0)}{y(x_0)} - h \int_0^x \frac{d\xi}{y^3(\xi)} \right] \right\}, \quad y \neq 0.$$

The constants  $x_0$  and  $h$  are specified in the next section. With  $f(x) = x - \beta x^3$  the above integral is an hyperelliptic integral of the first kind [3] which cannot be solved in closed form.

#### 4.2 Switching Condition

To show the existence or non-existence of a second switching one must investigate equation (7). This is done the following way. The initial conditions are chosen on the switching locus in the second quadrant for which  $u = -1$ , see figure 14. The integration is performed along the trajectory below the switching locus. Thus, from now on only the case  $u = +1$  is considered. By the symmetry of  $f(x)$  the situation is exactly the same above the switching locus with opposite sign. Let  $x_0 = -a$ , then the initial ordinate is

$$y(-a) = \sqrt{2a - a^2 + \frac{\beta}{2} a^4}.$$

Since  $(-a, y(-a))$  is on the switching locus  $\eta_2(-a) = 0$ . One can choose  $\eta_1(-a)$  arbitrarily because system (2) is homogeneous.  $\eta_1(-a)$  must be positive for  $u$  switches from  $-1$  to  $+1$  at this initial point. Thus let  $\eta_1(-a) = 1$  then by equation (6)

$$h = y(-a)$$

There exists a second switching when the integral in equation (7) becomes zero for some  $x \neq -a$ .

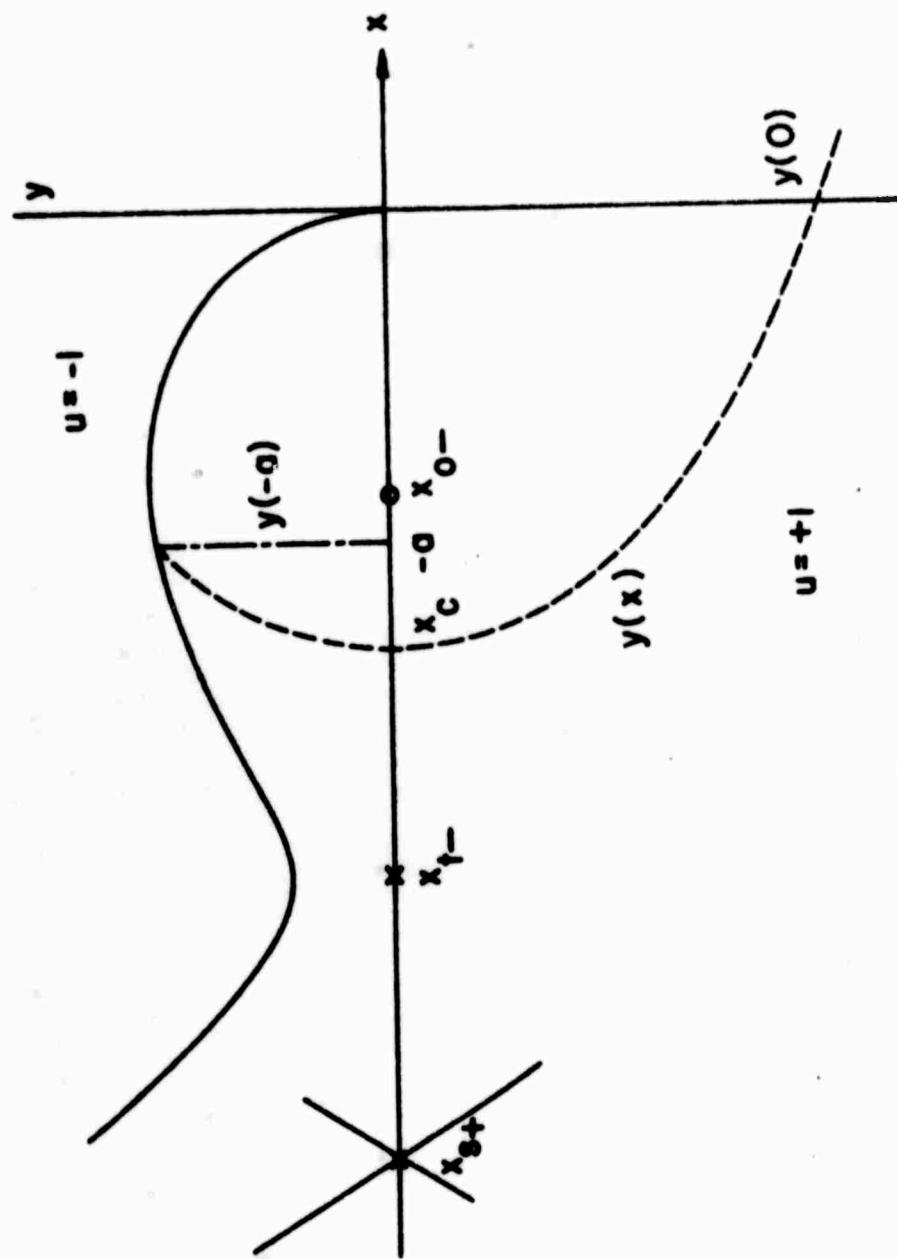


FIGURE 14. ILLUSTRATION OF THE INTEGRATION PROCEDURE.

However, as one can see from figure 1<sup>4</sup>,  $y(x)$  is not single-valued, so that the integration interval must be divided into subintervals for which  $y(x)$  is single-valued. The first interval is  $[-a, x_c]$  for which  $y(x) \neq 0$ . The second interval is  $(x_c, x]$  for which  $y(x)$  is in the lower half plane. The required division by  $y(x_c)$  causes difficulties. Later on in the evaluation of the integral it will be shown how to avoid this problem.

From (6) one can determine  $\eta_2(x_c)$

$$h = \eta_1(x_c) y(x_c) + \eta_2(x_c) [1 - f(x_c)] .$$

Since  $y(x_c) = 0$

$$\eta_2(x_c) = \frac{h}{1-f(x_c)} , \quad (8)$$

$\eta_2(x_c) > 0$  since  $1 - f(x_c) > 0$  for  $x_{s+} < x_c < 0$ .

It will be shown later that the next switching - if there exists one - occurs in the fourth quadrant.

Therefore, first  $\frac{\eta_2(0)}{y(0)}$  will be determined, and then the equation,

$$\frac{\eta_2(x)}{y(x)} = \frac{\eta_2(0)}{y(0)} - h \int_0^x \frac{d\xi}{y^3(\xi)} , \quad x > 0$$

will be examined. At a second switching point  $\eta_2(x) = 0$ , and

$$\frac{\eta_2(0)}{y(0)} - h \int_0^x \frac{d\xi}{y^3(\xi)} = 0$$

solved for  $x$  will give a point  $(x, y)$  of the switching locus. In general this equation cannot be solved. But if there is a switching for some  $0 < x < \infty$  the expression on the left hand side of this equation will change its sign. Since  $\eta_2(x) > 0$  and  $y(x) < 0$  in the third quadrant,

$\frac{\eta_2(0)}{y(0)} < 0$ . Thus, there exists a switching on  $(0, \infty)$  if the expression

$$\frac{\eta_2(0)}{y(0)} - h \int_0^\infty \frac{dx}{|y^3(x)|}$$

is positive, or if

$$\int_0^\infty \frac{dx}{|y^3(x)|} < \frac{\eta_2(0)}{h|y(0)|} .$$

Considering absolute values of  $y$  the switching condition is given by

$$\int_0^\infty \frac{dx}{|y^3(x)|} > \frac{\eta_2(0)}{h|y(0)|} . \quad (9)$$

If  $\int_0^\infty \frac{dx}{|y^3(x)|} < \frac{\eta_2(0)}{h|y(0)|}$ , (10)

there is no switching.

#### 4.3 Estimation

In this section estimates are derived to examine the inequalities (9) and (10).  $\frac{\eta_2(0)}{y(0)}$  is determined by

$$\frac{\eta_2(0)}{y(0)} = \lim_{x \rightarrow x_c^+} \left[ \frac{\eta_2(x)}{y(x)} - h \int_x^0 \frac{d\xi}{y^3(\xi)} \right], \quad (11)$$

where the integration is performed in the third quadrant. The above integral is subdivided into two integrals as follows

$$\int_x^0 \frac{d\xi}{y^3(\xi)} = \int_x^{-a} \frac{d\xi}{y^3(\xi)} + \int_{-a}^0 \frac{d\xi}{y^3(\xi)} .$$

Furthermore with  $y' = 1 - f(x)$  the first integral of the right hand side is transformed as follows

$$\int_x^{-a} \frac{d\xi}{y^3(\xi)} = \int_{y(x)}^{y(-a)} \frac{dy(\xi)}{y^2(\xi)[1-f(\xi)]} .$$

Thus, the equation to be investigated is now

$$\frac{y_2(0)}{y(0)} = \lim_{x \rightarrow x_c^+} \left[ \frac{y_2(x)}{y(x)} - h \int_{y(x)}^{y(-a)} \frac{dy}{y^{1-f}} \right] - h \int_{-a}^0 \frac{ds}{y^f(s)}.$$

For the estimation of  $1 - f(x)$  one needs to distinguish three different cases, as is shown in figure 15. At  $x = -\frac{1}{\sqrt{3\beta}}$   $1 - f(x)$  has its maximum value.

Case 1:  $x_{s+} < x_c < -\frac{1}{\sqrt{3\beta}} < -a < 0$

then  $1 - f(x) > 1 - f(-a)$  on  $(-\frac{1}{\sqrt{3\beta}}, -a)$

$1 - f(x) > 1 - f(x_c)$  on  $(x_c, -\frac{1}{\sqrt{3\beta}})$

This requires that  $y(-\frac{1}{\sqrt{3\beta}})$  exists:

$$y\left(-\frac{1}{\sqrt{3\beta}}\right) = \sqrt{\frac{4a}{18\beta} + \frac{5}{18\beta} - \frac{2}{\sqrt{3\beta}}}$$

thus, the condition on  $a$  is

$$\frac{1}{4}\left(\frac{5}{18\beta} + \frac{2}{\sqrt{3\beta}}\right) < a < \frac{1}{\sqrt{3\beta}}$$

or

$$a = \frac{1}{4}\left(\frac{2}{\sqrt{3\beta}} + \frac{5}{18\beta}\right) + \frac{\gamma}{4}\left(\frac{2}{\sqrt{3\beta}} - \frac{5}{18\beta}\right) \text{ with } 0 < \gamma \leq 1. \quad (12)$$

Case 2:  $x_{s+} < x_c < -a < -\frac{1}{\sqrt{3\beta}}$

then  $1 - f(x) > 1 - f(x_c)$  on  $(x_c, -a)$ .

Therefore,  $a$  must be such that  $|x_c| < |x_s|$  exists. Thus

$$\frac{1}{\sqrt{3\beta}} < a < \frac{|x_s|}{6} [ |x_s| + 3 ].$$

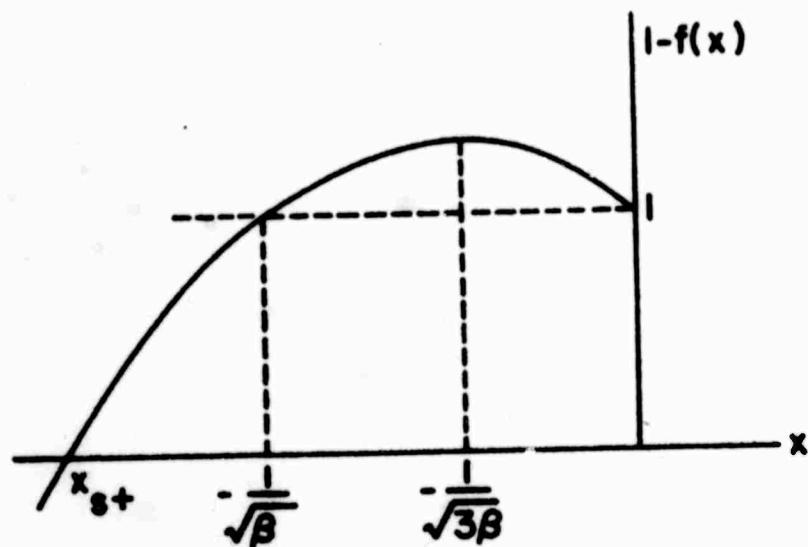


FIGURE 15. ILLUSTRATION OF  $1-f(x) = 1-x + \beta x^3$ .

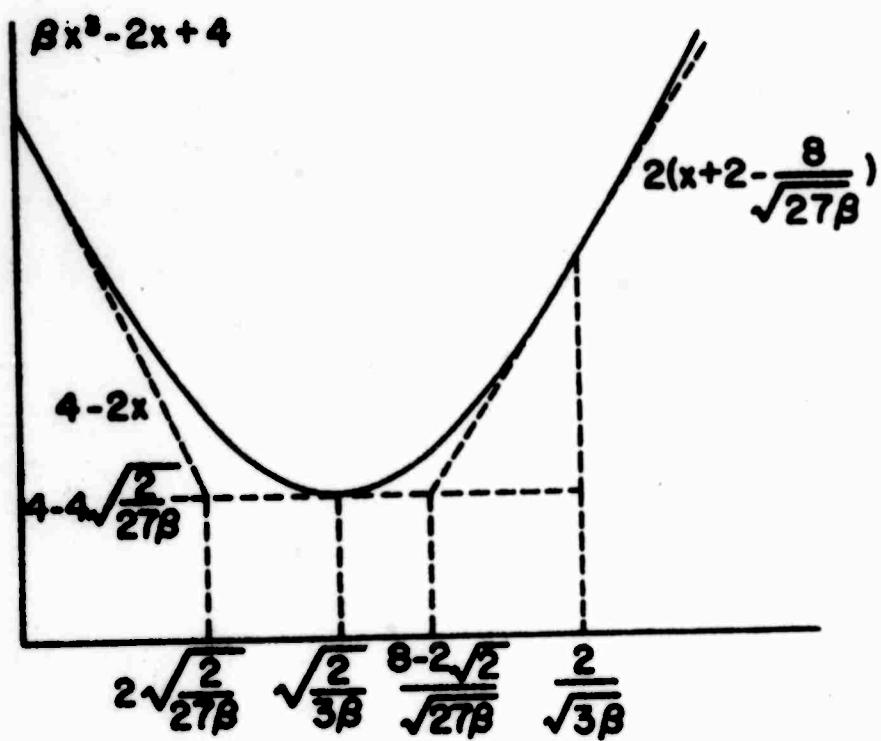


FIGURE 16. ILLUSTRATION OF THE FUNCTION  $\beta x^3 - 2x + 4$ .

$$\text{Case 3: } -\frac{1}{\sqrt{3B}} < x_c < -a < 0$$

then  $1 - f(x) < 1 - f(x_c)$  on  $(x_c, -a)$ .

The limits on  $a$  are then

$$c < a < \frac{1}{4} \left( \frac{2}{\sqrt{3B}} - \frac{5}{18B} \right).$$

For case 1 the integration is performed over the intervals  $[y(x), y(-\frac{1}{\sqrt{3B}})]$  and  $[y(-\frac{1}{\sqrt{3B}}), y(-a)]$ . Then the estimate is

$$\begin{aligned} \frac{\eta_2(0)}{y(0)} &< \lim_{x \rightarrow x_c^+} \left\{ \frac{\eta_2(x)}{y(x)} - \frac{h}{1-f(x_c)} \left[ \frac{1}{y(x)} - \frac{1}{y(-\frac{1}{\sqrt{3B}})} \right] \right\} \\ &- \frac{h}{1-f(-a)} \left[ \frac{1}{y(-\frac{1}{\sqrt{3B}})} - \frac{1}{y(-a)} \right] - h \int_{-a}^0 \frac{dx}{y^3(x)}. \end{aligned}$$

After performing the limit and taking absolute values for  $y$

$$\begin{aligned} \frac{\eta_2(0)}{h|y(0)|} &> \int_{-a}^0 \frac{dx}{y^3(x)} + \frac{\eta_1(x_c)}{h[1-f(x_c)]} \\ &+ \frac{1}{[1-f(x_c)] |y(-\frac{1}{\sqrt{3B}})|} \\ &- \frac{1}{1-f(-a)} \left[ \frac{1}{|y(-\frac{1}{\sqrt{3B}})|} - \frac{1}{h} \right] \quad (13) \end{aligned}$$

This holds for case 1. The estimates for cases 2 and 3 are given later.

In the above inequality (13) the integral and  $\eta_1(x_c)$  must be estimated. The equation for  $\eta_1(x)$  can easily be found from equation (7) and equation (6).

$$\frac{\eta_1(x)}{1-f(x)} = \frac{\eta_1(x_0)}{1-f(x_0)} - \frac{h}{y(x_0) [1-f(x_0)]}$$

$$+ \frac{h}{y(x) [1-f(x)]} + h \int_{y(x_0)}^{y(x)} \frac{dy}{y^2 [1-f]}$$

Starting at  $x_0 = -a$  on the switching locus the integration is performed on the trajectory in the second quadrant. Following the above estimation procedure one obtains for case 1

$$\frac{\eta_1(x_c)}{1-f(x_c)} > \frac{h}{[1-f(x_c)] |y(-\frac{1}{\sqrt{3b}})|}$$

$$+ \frac{h}{1-f(-a)} \left[ \frac{1}{h} - \frac{1}{|y(-\frac{1}{\sqrt{3b}})|} \right].$$

In the same way one obtains

$$\text{for case 2 } \eta_1(x_c) > 1$$

$$\text{for case 3 } \eta_1(x_c) < 1.$$

In the following estimates are pursued which show that there is no second switching, by inequality (10), for cases 1 and 2.

For the integral in (13)

$$\int_{-a}^0 \frac{dx}{y^3(x)} = - \int_{-a}^0 \frac{dx}{\left[ 4a + \frac{x}{2} (bx^3 - 2x + 4) \right]^{3/2}},$$

the cubic function  $bx^3 - 2x + 4$  is replaced by its tangent at  $x = 0$   
which is  $4-2x$ . Then the integral can be estimated

$$\int_{-a}^0 \frac{dx}{y^3(x)} > \frac{1}{1+4a} \left[ \frac{1}{2/a} - \frac{1+a}{\sqrt{2a-a^2}} \right].$$

Finally, the desired estimate for case 1 is

$$\begin{aligned}
 \frac{\pi_2(0)}{h|y(0)|} &> \frac{1}{1+4a} \left[ \frac{1}{2\sqrt{a}} - \frac{1+a}{\sqrt{2a-a^2}} \right] \\
 &+ \frac{2}{1-f(-a)} \left[ \frac{1}{h} - \frac{1}{|y(-\frac{1}{\sqrt{3B}})|} \right] \\
 &+ \frac{2}{1-f(-a)} \left[ \frac{1}{h} - \frac{1}{|y(-\frac{1}{\sqrt{3B}})|} \right]
 \end{aligned} \tag{14}$$

and for case 2

$$\begin{aligned}
 \frac{\pi_2(0)}{h|y(0)|} &> \frac{1}{1+4a} \left[ \frac{1}{2\sqrt{a}} - \frac{1+a}{\sqrt{2a-a^2}} \right] \\
 &+ \frac{h^2}{h|1-f(x_c)|} \tag{15}
 \end{aligned}$$

The estimate for case 3 is given on page 49.

Now one needs an estimate for the left side of inequality (10).

$$\int_{-\infty}^{\infty} \frac{dx}{|y^3(x)|} = \int_{-\infty}^{\infty} \frac{dx}{\left[ \frac{x}{2} (3x^2 - 2x - 4) + 4a \right]^{3/2}}$$

The cubic polynomial is approximated by its tangents is illustrated in

figure 16. Then

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{dx}{|y^3(x)|} &< \int_{-\infty}^{\sqrt{\frac{8}{27B}}} \frac{dx}{\left[ 4a + 1 - (x-1)^2 \right]^{3/2}} + \int_{\sqrt{\frac{8}{27B}}}^{\frac{8-2\sqrt{2}}{\sqrt{27B}}} \frac{dx}{\left[ 4a + 2(1 - \sqrt{\frac{2}{27B}})x + x^2 \right]^{3/2}} \\
 &+ \int_{\frac{8-2\sqrt{2}}{\sqrt{27B}}}^{\infty} \frac{dx}{\left[ 4a + 2(1 - \frac{4}{\sqrt{27B}})x + x^2 \right]^{3/2}}
 \end{aligned}$$

After performing these integrations one obtains the following inequality as equivalent to the condition that no second switching exists in case 1.

$$\begin{aligned}
 F(a, \beta) = & \left\{ \frac{1}{1+4a} \left[ \frac{1}{2\sqrt{a}} + \frac{\sqrt{\frac{8}{27\beta} - 1}}{\sqrt{4a + 4(\sqrt{\frac{2}{27\beta}} - \frac{2}{27\beta})}} \right] \right. \\
 & + \frac{1}{1 - \sqrt{\frac{2}{27\beta}}} \left[ \frac{1}{\sqrt{4a + 2\sqrt{\frac{2}{27\beta}} - \frac{2}{27\beta}}} - \frac{1}{\sqrt{4a + 4(2\sqrt{2}-1)(\sqrt{\frac{2}{27\beta}} - \frac{2}{27\beta})}} \right] \\
 & + \left. \frac{1}{4a - (1 - \frac{4}{\sqrt{27\beta}})^2} \left[ 1 - \frac{\sqrt{\frac{2}{27\beta}}(2\sqrt{2}-2) + 1}{\sqrt{4a + 4(2\sqrt{2}-1)(\sqrt{\frac{2}{27\beta}} - \frac{2}{27\beta})}} \right] \right\} \\
 & - \left\{ \frac{1}{1+4a} \left[ \frac{1}{2\sqrt{a}} - \frac{1+a}{\sqrt{2a-a}} \right] + \frac{2}{h [1 - f(-a)]} \right. \\
 & + \left. \frac{2}{4a - \frac{5}{18\beta} - \frac{2}{\sqrt{3\beta}}} \left[ \frac{1}{1 - f(x_c)} - \frac{1}{1 - f(-a)} \right] \right\} < 0 \quad (16)
 \end{aligned}$$

The second braces contain the quantity obtained in inequality (14). It must be replaced by the one from inequality (15) if case 2 is examined. All square roots in the first braces exist for  $\beta > \frac{2}{27}$ . The function  $-F(a, \beta)$  is plotted versus  $\beta$  in figure 17 with  $\gamma$  as parameter.  $-F(a, \beta)$  lies in the shaded area for  $0.2 \leq \gamma \leq 1.0$  and  $\beta \geq \frac{4}{27}$ . This shows that inequality (16) holds, i.e., that there is no second switching for  $\beta > \frac{4}{27}$  and  $\frac{1}{4} (\frac{5}{18\beta} + \frac{2}{\sqrt{3\beta}}) < a \leq \frac{1}{\sqrt{3\beta}}$ .

For case 2 a positive term in  $F(a, \beta)$  drops out, so that the approximation becomes even better.

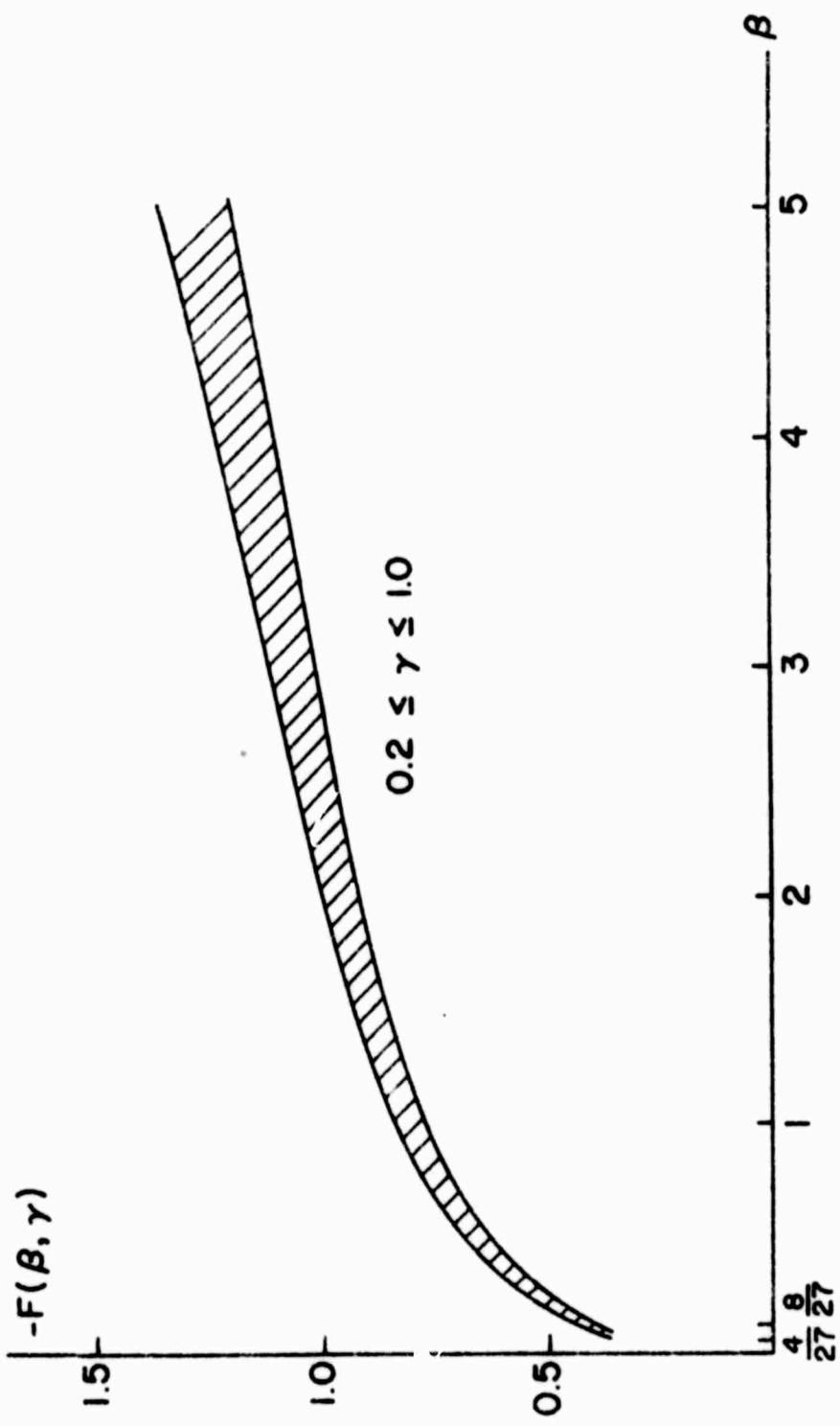


FIGURE 17. ILLUSTRATION OF INEQUALITY (16).

To show the existence of a second switching for some  $\beta > \frac{2}{27}$  and some  $a$  inequality (9) must be verified. Since there is apparently only a small range of  $a$  and  $\beta$  for which a second switching may exist (see figures 12 and 13), and since the estimates are not very sharp one example will be tested, which will be  $a = 0.2$ ,  $\beta = 0.06$ , see figure 12. One can see that case 3 applies:  $-\frac{1}{\sqrt{3\beta}} < x_c < -a$ , for which

$$\frac{\eta_2(0)}{h|y(0)|} < \frac{2}{h[1-f(x_c)]} + \int_{-a}^0 \frac{dx}{|y^3(x)|}.$$

There the cubic function  $4-2x+\beta x^3$  is replaced by its segments over 8 intervals on  $-a < x < 10$ . After the evaluation of all the estimates one obtains

$$\int_0^\infty \frac{dx}{|y^3(x)|} > 2.9$$

and

$$\frac{\eta_2(0)}{h|y(0)|} < 2.4$$

satisfying inequality (9). Thus there exists a second switching point. Because  $d\eta_2/dx$  exists and is nonzero if  $\eta_2 = 0$  there must be an interval about  $a = 0.2$  for each point of which a second switching point exists. The same can be said for  $\beta$ .

#### 4.4 Estimation for Small $a$

To prove the results in figures 12 and 13 one must show that there is no second switching for small  $a$ . A different approach is used to verify inequality (10).  $x$  is expressed as a function of  $y$ , and the integration is performed with respect to  $y$ . The requirement of  $a$  being small implies that the range of  $x$  and  $y$  is small in the

left half phase plane. For the equation describing the trajectory,

$$y^2 = 4a + 2x - x^2 + \frac{\beta}{2} x^4 ,$$

one can apply an approximation procedure to obtain  $x$  as a function of  $y$

$$x_k = 1 - \sqrt{r^2 - y^2 + \frac{\beta}{2} x_{k-1}^4} ,$$

where  $r^2 = 1 + 4a$ . This sequence converges for

$$\frac{\beta}{2} x^4 < r^2 - y^2 .$$

This approximation results in

$$x = x_c + A y^2 + B y^4 + C y^6 + \dots$$

$$\text{where } A = \frac{1}{2r} - \frac{\beta}{2} \frac{(1-r)^3}{r^2} - \frac{\beta}{8} \frac{(1-r)^4}{r^3} + \frac{\beta^2}{8} \frac{(1-r)^7}{r^4} + \dots$$

$$B = \frac{1}{8r^3} - \frac{3\beta}{8} \frac{(1-r)^2}{r^3} - \frac{3\beta}{8} \frac{(1-r)^3}{r^4} - \frac{3\beta}{32} \frac{(1-r)^4}{r^5} + \dots$$

$$C = \frac{5}{32r^5} - \frac{\beta}{8} \frac{1-r}{r^4} - \frac{3\beta}{8} \frac{(1-r)^2}{r^5} + \dots$$

$$\text{and } x_c = (1-r) - \frac{\beta}{4} \frac{(1-r)^4}{r} + \frac{\beta^2}{32} \frac{(1-r)^8}{r^3} + \dots$$

$$= -2a(1 - a + 2a^2 + 2\beta a^3 + \dots) .$$

The higher terms have coefficients with decreasing magnitude. The

above expressions are evaluated by developing series in  $a$  since

$r = \sqrt{1 + 4a}$ , whose power series converges for  $4a < 1$ . The following estimation is only valid for sufficiently small  $a$ . The integration in the second quadrant is then

$$\begin{aligned}\frac{\eta_2(x)}{y(x)} &= -h \int_{-a}^x \frac{d\xi}{y^3(\xi)} \\ &= -\left[ \frac{x_c + a}{y^3(-a)} - \frac{x_c - x}{y^3(x)} - 3 \int_{y(-a)}^{y(x)} \frac{x_c - \xi}{y^4} dy \right].\end{aligned}$$

After substituting the polynomials in  $y$  and performing the integration one obtains

$$\frac{\eta_2(x)}{y(x)} = \frac{2Ah}{y(x)} - 2A + 4Bh^2 - 2Ch^4 - 4By(x) - 2Chy^2(x).$$

Letting  $x$  approach  $x_c$

$$\lim_{x \rightarrow x_c^+} \frac{1}{y(x)} [\eta_2(x) - 2Ah] = -2A + 4Bh^2 - 2Ch^4. \quad (17)$$

Thus,  $\eta_2(x_c) = 2Ah = h \left[ \frac{1}{r} - \beta \frac{(1-r)^3}{r^2} - \frac{\beta}{4} \frac{(1-r)^4}{r^3} + \dots \right]$  which compares with  $\eta_2(x) = \frac{h}{1-r(x_c)}$  in the first three terms of its series in  $a$ .

In the third quadrant the integration is

$$\begin{aligned}\frac{\eta_2(0)}{y(0)} &= \frac{\eta_2(x)}{y(x)} - h \int_x^0 \frac{d\xi}{y^3(\xi)} \\ &= \frac{\eta_2(x)}{y(x)} - h \left[ \frac{x_c - x}{y^3(x)} - \frac{x_c}{y^3(0)} - 3 \int_{y(x)}^{y(0)} \frac{x_c - \xi}{y^4} dy \right].\end{aligned}$$

With the above polynomials and equation (17) one obtains the following estimate

$$\frac{\eta_2(0)}{h|y(0)|} = \frac{2A}{h} \left[ 1 + \frac{h}{\sqrt{4a}} \right] - 4B \sqrt{4a} \left[ 1 + \frac{h}{\sqrt{4a}} \right] - 2C(4a)^{\frac{3}{2}} \left[ 1 + \frac{h^3}{(4a)^{3/2}} \right] + \dots$$

or as a function of  $a$

$$\frac{y_2(0)}{h[y(0)]} \geq \frac{1}{\sqrt{2a}} \left[ 1 + \frac{1}{\sqrt{2}} - 5.29a + 10a^2 + \dots \right].$$

For the integral in the fourth quadrant one can use the estimate for case 1. Then the condition that there is no second switching for small  $a$  is

$$\begin{aligned} & \frac{1}{1+4a} \frac{\frac{2}{\sqrt{27\beta}} - 1}{2\sqrt{a} + \sqrt{\frac{2}{27\beta}} - \frac{2}{27\beta}} \\ & + \frac{1}{1 - \sqrt{\frac{2}{27\beta}}} \left[ \frac{1}{\sqrt{4a} - 2\sqrt{\frac{2}{27\beta}} - \frac{2}{27\beta}} - \frac{1}{\sqrt{4a} + 7.3(\sqrt{\frac{2}{27\beta}} - \frac{2}{27\beta})} \right] \\ & + \frac{1}{4a - (1 - \frac{4}{\sqrt{27\beta}})} \left[ 1 - \frac{0.83 \sqrt{\frac{2}{27\beta}} + 1}{\sqrt{4a} + 7.3(\sqrt{\frac{2}{27\beta}} - \frac{2}{27\beta})} \right] \\ & < \frac{1}{\sqrt{2a}} (1 - 2.46a - 1.3a^2 + \dots) \quad (18) \end{aligned}$$

This inequality is illustrated in figure 18. The left hand side of inequality (18) is plotted versus  $a$  with  $\beta$  as parameter. The right hand side marks the boundary of the region for each pair  $(a, \beta)$  of which the inequality holds. The inequality was also tested for  $\beta = 0.09$  and was satisfied for  $a \leq 0.1$ . In the preceding section it was shown that there is no second switching for  $a \geq \frac{1}{4} (\frac{2}{\sqrt{3\beta}} + \frac{5}{18\beta})$ . For  $\beta \geq 2$  (18) holds for all  $a$  less than the above value. This implies that there is no second switching for  $\beta \geq 2$ . There does not exist a second switching for any  $\beta > \frac{2}{27}$  and sufficiently small  $a$ .

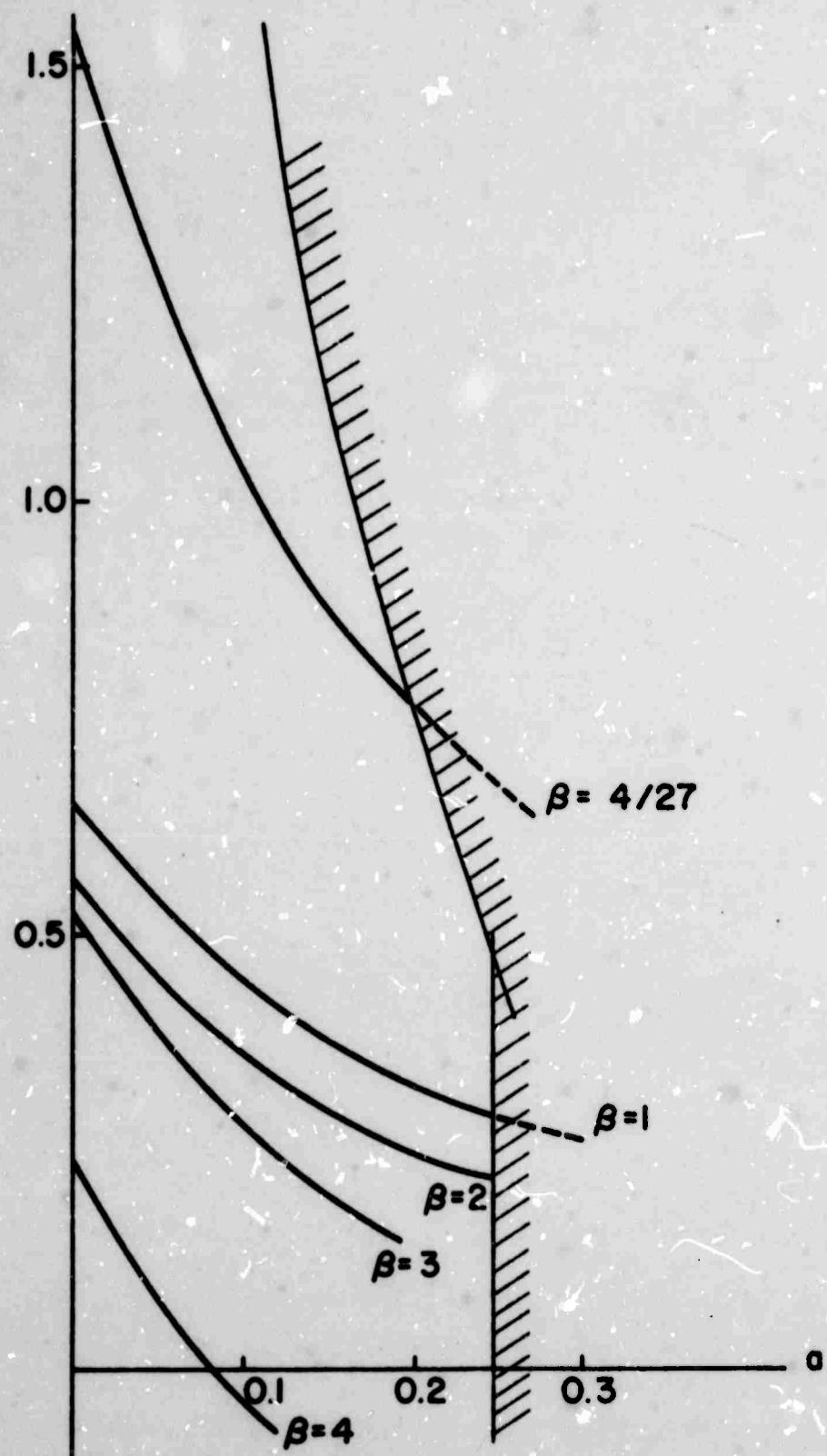


FIGURE 18. ILLUSTRATION OF INEQUALITY (18).

#### 4.5 Non-Existence of Switchings in the Third Quadrant

The trajectory leaving the switching locus through the origin in the second quadrant backward in time is given by

$$y_+(x) = \pm \sqrt{4a + 2x - x^2 + \frac{B}{2} x^4}$$

Assume  $\tau_2 = 0$  in the third quadrant at  $x_1$ . Then  $u$  switches from +1 to -1 and the trajectory is now described by

$$y_-(x) = \pm \sqrt{4(a + x_1) - 2x - x^2 + \frac{B}{2} x^4}.$$

If  $x_1 < -a$ ,  $y_-(x)$  reintersects the negative  $x$ -axis entering the second quadrant again, since it cannot intersect the  $y$ -axis:  $y_-(0) = \sqrt{4a + x_1}$  is imaginary. Thus, this is not a time-optimal process. The trajectory is now closer to the origin than when it started going backwards. If  $-a < x_1$  the trajectory  $y_-(x)$  intersects the  $y$ -axis at

$$y_-(0) = -\sqrt{4a + x_1} > -\sqrt{4a},$$

where  $-\sqrt{4a} = y_+(0)$ . As the trajectory  $y_-(x)$  enters the fourth quadrant continuously increasing it must intersect the switching locus through the origin in this quadrant at  $x_2$ ,

$$x_2 = a + x_1.$$

with  $-a < x_1 < 0$ :  $0 < x_2 < a$ .

A trajectory coming from above this switching locus in forward time reaches the origin more quickly along this switching locus than along the prescribed way. It takes even longer to reach the origin from the starting point  $(-a, y(-a))$  than from  $(x_2, -y(x_2))$ . Thus, there cannot be a switching in the third quadrant for a time-optimal control.

CHAPTER V  
SUMMARY AND RECOMMENDATIONS

5.1 Summary

In this investigation results for the synthesis problem were obtained. It was shown that the coefficient  $\beta > 0$  of the negative nonlinear term of the Duffing equation characterizes the behavior of the system.

For  $0 \leq \beta \leq \frac{2}{27}$  the maximum number of switchings  $k$  is  $\geq k \geq 2$ .

A procedure was developed to divide the domain of controllability  $G$  into subregions for each point of which a certain number of switchings is necessary to steer it to the origin time-optimally. With this procedure one also obtains the maximum number of switchings.

Computer results opposed the assumption that for  $\beta > \frac{2}{27}$  the trajectory through the origin in the second and fourth quadrants, coincides with the switching locus over its entire length, implying that  $k = 1$ . Namely, arcs of the switching locus sticking back from infinity were discovered for values  $\beta > \frac{2}{27}$ . Thus a switching condition involving functions of  $x$  and  $y$  was derived. Since it contains a hyperelliptic integral it was necessary to derive estimates to examine this condition. Using these estimates it was shown that there exists a second switching for an interval about  $\beta = 0.09$  and about  $a = 0.2$ .  $a$  is the abscissa of the point where the trajectory intersects the switching locus through the origin. It was also shown that

there does not exist a second switching for sufficiently small  $a$  and any  $\beta > \frac{2}{27}$ , and if  $\beta \geq \frac{4}{27}$  there does not exist a second switching if  $a > \frac{1}{4} (\frac{2}{\sqrt{3}\beta} + \frac{5}{18\beta})$ .

### 5.2 Recommendations

With these estimates one could not cover the whole range of  $a$  to determine whether there exists a second switching or not for

$\frac{4}{27} < \beta < 2$ . Thus more investigation on this hyperelliptic integral, equation (7), is necessary to obtain more information about the arc of the switching locus sticking back from infinity.

Another problem for possible future study is the case when the rate of the time-optimal control is bounded. The system is then the following

$$\dot{x} = y$$

$$\dot{y} = -f(x) + z$$

$$\dot{z} = v \quad \text{with } |v| \leq 1,$$

$v = \dot{u}$  is the rate of the control  $u$ . Thus a third state variable  $z$  is introduced, and the new control  $v$  is bounded in magnitude. The adjoint system is then

$$\dot{\eta}_1 = f'(x) \eta_2$$

$$\dot{\eta}_2 = -\eta_1$$

$$\dot{\eta}_3 = -\eta_2$$

and the Hamiltonian is

$$H = \eta_1 y + \eta_2 [z - f(x)] + \eta_3 v = h = \text{constant}.$$

In figure 19 an example for  $\beta = 0.03$  is shown. The integration is

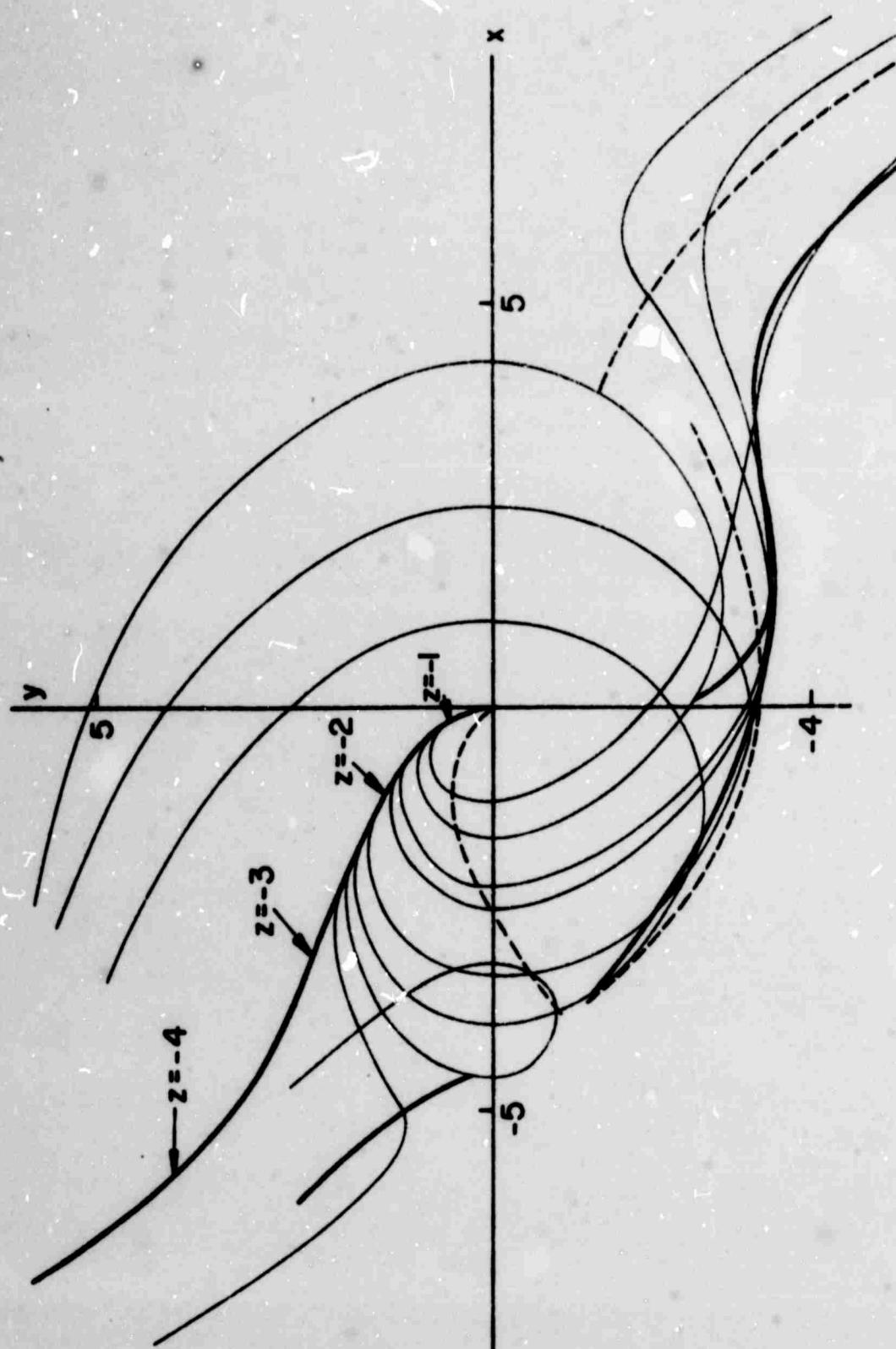


FIGURE 19. EXAMPLE FOR BOUNDED RATE OF CONTROL,  $\beta = 0.03$ .

performed backward in time starting in the origin with the following initial conditions:  $v = +1$  (the case  $v = -1$  is symmetric starting in the fourth quadrant) and  $\eta_3 = +1$ . Then  $h = 1$ . Since the adjoint system is homogeneous one variable  $\eta_1$  or  $\eta_2$  can be chosen arbitrarily, then the other variable is determined by the Hamiltonian. In this integration the initial value of  $\eta_2$  was varied for each trajectory. In figure 19 the heavy lines represent the switching locus where  $v$  changes its sign. The dashed lines mark the points  $(x,y)$  where  $z = 0$ . Much work has been done for linear systems with bounded rate of control, e.g., [7]. But for the nonlinear system the domain of controllability needs to be determined and the set of attainability should be described. An investigation on the maximum number of switchings would be of interest.

A further problem is to have a constraint on both the magnitude and the rate of the time-optimal control. Then for the above system the control  $v$  and the new state variable  $z$  are bounded. Some results are available for linear systems, e.g., [8]. But nothing has been done yet for nonlinear systems.

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